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If $a,b,c>0$ then:

$$a\cdot 2^{\frac{1}{a}}+b\cdot 2^{\frac{1}{b}}+c\cdot 2^{\frac{1}{c}}\geq (a+b+c)\cdot 8^{\frac{1}{a+b+c}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Using $AM\geq GM$

$$\frac{a\left(2^{\frac{1}{a}}\right)+b\left(2^{\frac{1}{b}}\right)+c\left(2^{\frac{1}{c}}\right)}{a+b+c}\geq\left[\left(2^{\frac{1}{a}}\right)^{a}\left(2^{\frac{1}{b}}\right)^{b}\left(2^{\frac{1}{c}}\right)^{c}\right]^{\frac{1}{a+b+c}}$$

$$\Rightarrow a\left(2^{\frac{1}{a}}\right)+b\left(2^{\frac{1}{b}}\right)+c\left(2^{\frac{1}{c}}\right)\geq (a+b+c)(8)^{\frac{1}{a+b+c}}$$

Equality holds for $a=b=c$.

Solution 2 by Khaled Abd Imouti

$$f(x)=x\cdot 2^{\frac{1}{x}}=x\cdot e^{\frac{1}{x}\ln(2)},\quad f'(x)=e^{\frac{1}{x}\ln(2)}+\left(-\frac{x}{x^2}\ln(2)\right)\cdot e^{\frac{1}{x}\ln(2)}$$

$$f'(x)=\left(1-\frac{1}{x}\ln(2)\right)\cdot e^{\frac{1}{x}\ln(2)}$$

$$f''(x)=\frac{1}{x^2}\ln(2)\cdot e^{\frac{1}{x}\ln(2)}-\frac{1}{x^2}\ln(2)\,e^{\frac{1}{x}\ln(2)}\cdot\left(1-\frac{1}{x}\ln(2)\right)$$

$$f''(x)=\frac{1}{x^2}\ln(2)\cdot e^{\frac{1}{x}\ln(2)}\cdot\left[1-1+\frac{1}{x}\ln(2)\right]=\frac{1}{x^3\ln^2(2)\,e^{\frac{1}{x}\ln(2)}}$$

$f''(x)=\frac{1}{x^2}\ln^2(2)\cdot 2^{\frac{1}{x}}>0$. So f is convex function and then:

$$a\cdot 2^{\frac{1}{a}}+b\cdot 2^{\frac{1}{b}}+c\cdot 2^{\frac{1}{c}}\geq 3\cdot\left(\frac{a+b+c}{3}\right)\cdot 2^{\frac{1}{\left(\frac{a+b+c}{3}\right)}}$$

$$a\cdot 2^{\frac{1}{a}}+b\cdot 2^{\frac{1}{b}}+c\cdot 2^{\frac{1}{c}}\geq (a+b+c)\cdot(8)^{\frac{1}{a+b+c}}$$

Equality holds for $a=b=c$.