A generalization, a refinement and a reverse for Cesaro's inequality

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In this work a generalization of Cesaro's inequality are presented and demonstrated. Also – for this generalization is obtained a refinement and a reverse inequality.

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Is well known and often used the *Cesaro's inequality*,

$$(a+b)(b+c)(c+a) \geq 8\,a\,b\,c\ . \qquad \textbf{(1)}$$

The inequality in the following sentence was proposed in [1], then taken over and presented in the prestigious site *Cut-the-knot*, [2] :

1. *Proposition*

If $a, b, c \in (0, \infty)$, such that $a \neq b \neq c \neq a$ and $n \in \mathbb{N}^*$, then :

$$\frac{a^{n+1}-b^{n+1}}{a-b}\cdot\frac{b^{n+1}-c^{n+1}}{b-c}\cdot\frac{c^{n+1}-a^{n+1}}{c-a} > (n+1)^3\cdot(abc)^n\ . \qquad \textbf{(2)}$$

Proof

With the help of a well–known decomposition formula and then with the *inequality of means*, we have successively :

$$\frac{a^{n+1}-b^{n+1}}{a-b} = \sum_{k=0}^{n} a^{n-k}b^k \overset{\text{(AM}-\text{GM)}}{\geq} (n+1)\cdot\sqrt[n+1]{\prod_{k=0}^{n} a^{n-k}b^k} =$$

$$= (n+1)\cdot\sqrt[n+1]{\prod_{k=0}^{n}(ab)^{1+2+\ldots+n}} = (n+1)\cdot\sqrt[n+1]{\prod_{k=0}^{n}(ab)^{n(n+1)/2}} = (n+1)\cdot(ab)^{n/2}\ .$$

How $a \neq b$, the inequality is strict, therefore :

$$\frac{a^{n+1}-b^{n+1}}{a-b} > (n+1)\cdot(a\,b)^{n/2} \cdot \qquad (3)$$

Analogously we have: $$\frac{b^{n+1}-c^{n+1}}{b-c} > (n+1)\cdot(b\,c)^{n/2} , \qquad (4)$$

$$\frac{a^{n+1}-b^{n+1}}{a-b} > (n+1)\cdot(a\,b)^{n/2} \cdot \qquad (5)$$

By multiplying relationships (3) , (4) , (5) inequality (2) from the statement is obtained .

2. *Corollary* **(*Generalization of Cesaro's inequality*)**

If $a\,,b\,,c\in[0\,,\infty)$ and $n\in\mathbb{N}^*$, then :

$$\left(\sum_{k=0}^{n}a^{n-k}b^{k}\right)\cdot\left(\sum_{k=0}^{n}b^{n-k}c^{k}\right)\cdot\left(\sum_{k=0}^{n}c^{n-k}a^{k}\right) \geq (n+1)^3\cdot(a\,b\,c)^n \cdot \qquad (6)$$

The ***Proof*** is extracted from the one above - after the first equal, with the observation that here we also have the possibility of equality , when $a=b=c$.

3. *Remark*

For $n=1$ in inequality (6), *Cesaro's inequality* is obtained.
For $n=2$ in inequality (6), the following inequality is obtained ,

$$\left(a^2+ab+b^2\right)\left(b^2+bc+c^2\right)\left(c^2+ca+a^2\right)\geq 27\,(a\,b\,c)^2 \cdot \qquad (7)$$

Another ***Proof*** for Proposition 1 is obtained using the well-known inequality of *Hermite – Hadamard* :

- if f is a convex function on the interval $[a\,,b]$, then occurs the inequality ,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\cdot\int_a^b f(x)\cdot dx \leq \frac{f(a)+f(b)}{2} \cdot \qquad (8)$$

Considering the function $f(x)=x^n$, obviously convex and increasing on $[\,0\,,\infty\,)$, using the first inequality from (8) and the inequality of means , we successively have :

$$\frac{1}{b-a}\cdot\int_a^b x^n\cdot dx \geq \left(\frac{a+b}{2}\right)^n \Leftrightarrow \frac{b^{n+1}-a^{n+1}}{(n+1)\cdot(b-a)} \geq \left(\frac{a+b}{2}\right)^n \geq \left(\sqrt{a\,b}\right)^n \Leftrightarrow$$

$$\Leftrightarrow \frac{b^{n+1}-a^{n+1}}{b-a} \geq (n+1)\cdot\left(\sqrt{a\,b}\right)^n ,$$

that is , inequality (3) . Further, the proof proceeds as in the proof of *Proposition* 1 .

4. *Remark*

Grace to inequality (8) , by taking $f(x)=x^t$, t real number , $t>1$ - we can expand the inequality from relation (2) to real powers , too ; see also [2].

But using the *Hermite - Hadamard inequality* actually allows us to get more than the proof of inequality from Proposition 1. It also provides the following,

5. *Proposition* (***Refinement and reverse of generalized Cesaro's inequality***)

If $a, b, c \in [0,\infty)$, and $n \in \mathbb{N}^*$, then :

$$(n+1)^3 \cdot (abc)^n \le \frac{(n+1)^3}{8^n} \cdot (a+b)^n (b+c)^n (c+a)^n \le$$

$$\le \left(\sum_{k=0}^{n} a^{n-k} b^k\right) \cdot \left(\sum_{k=0}^{n} b^{n-k} c^k\right) \cdot \left(\sum_{k=0}^{n} c^{n-k} a^k\right) \le$$

$$\le \frac{(n+1)^3}{8} \cdot (a^n+b^n)(b^n+c^n)(c^n+a^n) \cdot \qquad (9)$$

The ***Proof*** is obtained using mainly the inequality (8) .

For $n \in \mathbb{N}_{\ge 2}$, equality occurs if and only if $a=b=c$.

6. *Remark*

For $n \in \mathbb{N}_{\ge 2}$, the second inequality in (9) constitutes a *refinement* of the *generalized inequality of Cesaro* (6) , and the last inequality in (9) is an *inverse inequality* of the *generalized Cesaro's inequality* (6) , thus establishing an inequality on the right side of the product $\left(\sum_{k=0}^{n} a^{n-k} b^k\right) \cdot \left(\sum_{k=0}^{n} b^{n-k} c^k\right) \cdot \left(\sum_{k=0}^{n} c^{n-k} a^k\right)$.

For $n=1$, all the inequalities in (9) - starting with the second one - become equalities. This is how inequality is presented (9) , for $n=2$:

7. *Corollary*

If $a, b, c \in [0,\infty)$, then holds the inequalities :

$$27(abc)^2 \le \frac{27}{64} \cdot (a+b)^2 (b+c)^2 (c+a)^2 \le$$

$$\le (a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)$$

$$\le \frac{27}{8} \cdot (a^2+b^2)(b^2+c^2)(c^2+a^2) \cdot \qquad (10)$$

It is to be expected that almost any inequality that is demonstrated with *Cesaro's inequality* to benefit from the results of this note . We reserve the right to revert to the possibilities of applying these results .

References

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