ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sum_{\substack{(n_1,n_2,n_3,n_4)\in\mathbb{Z}^4\\(n_1,n_2,n_3,n_4)\neq(0,0,0,0)}} \frac{1}{(n_1^2+n_2^2+n_3^2+n_4^2)^s} = 8\left(1-\frac{1}{4^{s-1}}\right)\zeta(s-1)\zeta(s)$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Shobhit Jain-India

$$\sum_{\substack{(n_1,n_2,n_3,n_4)\in\mathbb{Z}^4\\(n_1,n_2,n_3,n_4)\neq(0,0,0,0)}} \frac{1}{(n_1^2+n_2^2+n_3^2+n_4^2)^s} = \sum_{n\in\mathbb{N}} \frac{r_4(n)}{n^s}$$

$$(by\ using\ Jacobi's\ four\ square\ identity)$$

$$Now,\ \ r_4(n) = 8\sigma(n) - 32\sigma(n/4).\ \ Here,\ \ \sigma(n/4) = 0\ \ for\ \ n \not\equiv 0 (mod\ 4)$$

$$reference: Ramanujan's\ Notebook\ Part\ V\ Chapter\ 36,\ \ page\ 377$$

$$\Rightarrow \sum_{\substack{(n_1,n_2,n_3,n_4)\in\mathbb{Z}^4\\(n_1,n_2,n_3,n_4)\neq(0,0,0,0)}} \frac{1}{(n_1^2+n_2^2+n_3^2+n_4^2)^s} = 8\sum_{n\in\mathbb{N}}\frac{\sigma(n)}{n^s} - 32\sum_{n\in 4\mathbb{N}}\frac{\sigma\left(\frac{n}{4}\right)}{n^s} =$$

$$= 8\sum_{n\in\mathbb{N}}\frac{\sigma(n)}{n^s} - 32\sum_{m\in\mathbb{N}}\frac{\sigma(m)}{(4m)^s} = \left(8-\frac{32}{4^s}\right)\sum_{n\in\mathbb{N}}\frac{\sigma(n)}{n^s} = 8\left(1-\frac{4}{4^s}\right)\sum_{n\in\mathbb{N}}\frac{\sigma(n)}{n^s} =$$

$$= 8\left(1-\frac{1}{4^{s-1}}\right)\sum_{n\in\mathbb{N}}\frac{\sigma(n)}{n^s}$$

$$Now,\ \ \sigma(n) = \textstyle\sum_{d|n} d = (I * ¶)(n)\ \ here,\ \ I(n) = n\quad and\quad ¶(n) = 1\quad \forall\, n \in N$$

$$Note: \sigma, I, ¶\ are\ multiplicative\ functions$$

$$\Rightarrow \sum_{n\in\mathbb{N}}\frac{\sigma(n)}{n^s} = \sum_{n\in\mathbb{N}}\frac{(I*¶)(n)}{n^s} = \left(\sum_{n\in\mathbb{N}}\frac{I(n)}{n^s}\right)\left(\sum_{n\in\mathbb{N}}\frac{¶(n)}{n^s}\right) = \left(\sum_{n\in\mathbb{N}}\frac{n}{n^s}\right)\left(\sum_{n\in\mathbb{N}}\frac{1}{n^s}\right) = \zeta(s-1)\zeta(s)$$

$$\Rightarrow \sum_{\substack{(n_1,n_2,n_3,n_4)\in\mathbb{Z}^4\\(n_1,n_2,n_3,n_4)\neq(0,0,0,0)}} \frac{1}{(n_1^2+n_2^2+n_3^2+n_4^2)^s} = 8\left(1-\frac{1}{4^{s-1}}\right)\zeta(s-1)\zeta(s)$$