ROMANIAN MATHEMATICAL MAGAZINE

$$If\ a,b,c>0, a+b+c=1\ then$$

$$27\left(\sqrt[9]{a^{10}+b^{10}+c^{10}}-a^2-b^2\right)+1\geq 2(6c+\ln(3c))$$

Proposed by Pavlos Trifon-Greece

Solution by Tapas Das-India

$$Let\ f(t)=t^9, t>0\ so\ f\ is\ convex\ .Using\ Jensen's\ inequality\ we\ get$$

$$af(a)+bf(b)+cf(c)\geq(a+b+c)f\left(\frac{a^2+b^2+c^2}{a+b+c}\right)\overset{a+b+c=1}{=} f(a^2+b^2+c^2)$$

$$or, a.a^9+b.b^9+c.c^9=a^{10}+b^{10}+c^{10}=(a^2+b^2+c^2)^9$$

$$we\ need\ to\ show: 27\left(\sqrt[9]{a^{10}+b^{10}+c^{10}}-a^2-b^2\right)+1\geq 2(6c+\ln(3c))$$

$$or, 27(a^2+b^2+c^2-a^2-b^2)+1\geq 2(6c+\ln(3c))\ or, 27c^2-12c+1-2\ln 3c\geq 0$$

$$or, 3\,t^2-4t+1-2\ln t\overset{t=3c}{\geq} 0\ , let\ g(t)=3\,t^2-4t+1-2\ln t$$

$$now\ g'(t)=\frac{6t^2-4t-2}{t}=\frac{2(3t+1)(t-1)}{t}$$

$$For\ 0<t<1\ , g'(t)<0\ \&\ for\ t>1\ ,\qquad g'(t)>0$$

$$thus\ g(t) achieves\ its\ global\ minimum\ at\ t=1\ \&\ g(1)=0\ then\ g(t)\geq 0$$

$$or, 3\,t^2-4t+1-2\ln t\geq 0$$

$$Equality\ holds\ for\ a=b=c=\frac{1}{3}$$