ROMANIAN MATHEMATICAL MAGAZINE

If $n=2,3,\ldots$ & $a,b,c>0$ *with* $abc\geq a+b+c$ *then*:

$$\frac{2^{\frac{abc}{a+b+c}}+3^{\frac{abc}{a+b+c}}+\cdots+n^{\frac{abc}{a+b+c}}}{n-1}+\frac{n(abc)^2}{18}\geq 2n+1$$

Proposed by Pavlos Trifon-Greece

Solution by Tapas Das-India

$$Let\ x=\frac{abc}{a+b+c}\geq 1\ as\ abc\geq a+b+c$$

$$a+b+c\overset{AM-GM}{\geq}3\sqrt[3]{abc}$$

$$abc\geq a+b+c\ or, abc\geq 3\sqrt[3]{abc}\ or, abc\geq 3\sqrt{3}\ (A)$$

$$\frac{n(abc)^2}{18}\overset{by\ (A)}{\geq}\frac{27n}{18}=\frac{3n}{2}$$

$$Let\ f(t)=t^x, t>0\ \&\ x\geq 1\ , so\ f\ is\ convex\ for\ t>0$$

By Jensen's inequality:

$$\frac{2^x+3^x+\cdots+n^x}{n-1}\geq\left(\frac{2+3+\cdots+n}{n-1}\right)^x=\left(\frac{(n-1)(n+2)}{2(n-1)}\right)^x=\left(\frac{n+2}{2}\right)^x$$

$$Now\ n\geq 2\ then\frac{n+2}{2}\geq 2>1, x\geq 1\ so\ the\ value\ icreases\ as\ x\ increases$$

$$so\ \left(\frac{n+2}{2}\right)^x\geq\left(\frac{n+2}{2}\right)^1=\frac{n+2}{2}$$

$$L.H.S=\frac{2^{\frac{abc}{a+b+c}}+3^{\frac{abc}{a+b+c}}+\cdots+n^{\frac{abc}{a+b+c}}}{n-1}+\frac{n(abc)^2}{18}\geq\frac{n+2}{2}+\frac{3n}{2}=2n+1=R.H.S$$

$$Equality\ holds\ for\ a=b=c=\sqrt{3}.$$