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If $a, b > 0$ then:

$$\sqrt{\sqrt{16a^4 + 1} + \frac{128}{\sqrt{17}(b+1)^2}} + \sqrt{\sqrt{16b^4 + 1} + \frac{128}{\sqrt{17}(a+1)^2}} \geq 2\sqrt{\frac{49}{\sqrt{17}}}$$

Proposed by Pavlos Trifon-Greece

Solution by Tapas Das-India

Lemma: $\forall t > 0, \sqrt{16t^4 + 1} \geq \frac{16t^2 + 1}{\sqrt{17}}$

Proof: $\sqrt{16t^4 + 1} \geq \frac{16t^2 + 1}{\sqrt{17}}$ or, $17(16t^4 + 1) \geq (16t^2 + 1)^2$

$16t^4 - 32t^2 + 16 \geq 0$ or $16(t^2 - 1)^2 \geq 0$ true.

$$\begin{aligned} & \sqrt{\sqrt{16a^4 + 1} + \frac{128}{\sqrt{17}(b+1)^2}} + \sqrt{\sqrt{16b^4 + 1} + \frac{128}{\sqrt{17}(a+1)^2}} \geq \\ & \stackrel{\text{Lemma}}{\geq} \sqrt{\frac{16a^2 + 1}{\sqrt{17}} + \frac{128}{\sqrt{17}(b+1)^2}} + \sqrt{\frac{16b^2 + 1}{\sqrt{17}} + \frac{128}{\sqrt{17}(a+1)^2}} = \\ & = \frac{1}{\sqrt[4]{17}} \left(\sqrt{16a^2 + 1 + \frac{128}{(b+1)^2}} + \sqrt{16b^2 + 1 + \frac{128}{(a+1)^2}} \right) = \\ & = \frac{1}{\sqrt[4]{17}} \left(\sqrt{(4a)^2 + 1^2 + \left(\frac{\sqrt{128}}{b+1}\right)^2} + \sqrt{(4b)^2 + 1^2 + \left(\frac{\sqrt{128}}{a+1}\right)^2} \right) \stackrel{\text{Minkowski}}{\geq} \\ & \geq \frac{1}{\sqrt[4]{17}} \sqrt{(4a + 4b)^2 + (1 + 1)^2 + 128 \left(\frac{1^2}{a+1} + \frac{1^2}{b+1} \right)^2} \geq \\ & \stackrel{\text{Bergstrom}}{\geq} \frac{1}{\sqrt[4]{17}} \sqrt{(4a + 4b)^2 + (1 + 1)^2 + 128 \left(\frac{4}{a+b+2} \right)^2} \stackrel{s=a+b}{=} \\ & = \frac{1}{\sqrt[4]{17}} \sqrt{16s^2 + 4 + \frac{2048}{(s+2)^2}} = \frac{2}{\sqrt[4]{17}} \sqrt{4s^2 + 1 + \frac{512}{(s+2)^2}}. \text{ Now we need to show:} \\ & \frac{2}{\sqrt[4]{17}} \sqrt{4s^2 + 1 + \frac{512}{(s+2)^2}} \geq 2 \cdot \sqrt{\frac{49}{\sqrt{17}}} \\ & \text{or, } 4s^2 + 1 + \frac{512}{(s+2)^2} \geq 49 \text{ or, } 4s^2(s+2)^2 + (s+2)^2 + 512 \geq 49(s+2)^2 \\ & \text{or, } 4(s^4 + 4s^3 - 8s^2 - 48s + 80) \geq 0 \text{ or, } (s-2)^2(s^2 + 8s + 20) \geq 0 \text{ true} \\ & \text{Equality holds for } s=a+b=2 \text{ or, } a=b=1 \end{aligned}$$