ROMANIAN MATHEMATICAL MAGAZINE

Let $\alpha, b, c > 1$. Prove that:

$$\frac{1}{(1+\log_b \alpha)^5}+\frac{1}{(1+\log_c b)^5}+\frac{1}{(1+\log_\alpha c)^5} \geq 0.09375$$

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Let $\log_b \alpha = \frac{v}{u}$, $\log_c b = \frac{w}{v}$, and $\log_\alpha c = \frac{u}{w}$ for $u, v, w > 0$.

$$Here, \frac{1}{1+\log_b \alpha} = \frac{1}{1+\frac{v}{u}} = \frac{u}{u+v} \quad and\ 0.09375 = \frac{3}{32}.$$

Thus, it suffices to prove that:

$$\sum_{cyc}\left(\frac{u}{u+v}\right)^5 \geq \frac{3}{32}$$

Via the 5-th and 2-nd Power Means Inequality:

$$\sum_{cyc}\left(\frac{u}{u+v}\right)^5 \overset{\textcircled{1}}{\geq} 3\left[\frac{1}{3}\sum_{cyc}\left(\frac{u}{u+v}\right)^2\right]^{\frac{5}{2}}$$

Via the Cauchy-Schwarz Inequality:

$$\left[\sum_{cyc}\left(\frac{u}{u+v}\right)^2\right]\left[\sum_{cyc}(u+v)^2(u+w)^2\right] \geq \left[\sum_{cyc} u\,(u+w)\right]^2 = \frac{1}{4}\left[\sum_{cyc}(u+v)^2\right]^2$$

Via $(X+Y+Z)^2 \geq 3(XY+YZ+ZX)$, we have:

$$\sum_{cyc}(u+v)^2(u+w)^2 \leq \frac{1}{3}\left[\sum_{cyc}(u+v)^2\right]^2 \Rightarrow \sum_{cyc}\left(\frac{u}{u+v}\right)^2 \geq \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$$

Substitution of this result back into ① yields:

$$\sum_{cyc}\left(\frac{u}{u+v}\right)^5 \geq 3\left(\frac{1}{3}\cdot\frac{3}{4}\right)^{\frac{5}{2}} = 3\left(\frac{1}{4}\right)^{\frac{5}{2}} = \frac{3}{32} = 0.09375$$

Equality holds if and only if $u = v = w$, which implies $\alpha = b = c$.