ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c\in\left[0,\frac{\pi}{2}\right)$ then:

$$9^{\sin a}+81^{\sin b}+729^{\sin c}+3^{\tan a}+9^{\tan b}+27^{\tan c}\ge 2\sqrt{3^{a+2b+3c+2}}$$

Proposed by Pavlos Trifon-Greece

Solution by Tapas Das-India

$$We\ will\ prove\ that:$$

$$2sinx+tanx\ge 3x\ for\ x\in\left[0,\frac{\pi}{2}\right)\ (1)$$

$$Let\ f(x)=2sinx+tanx-3x\ then:$$

$$f'(x)=2cosx+\sec^2 x-3=cosx+cosx+\frac{1}{\cos^2 x}-3\overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{cosx.cosx.\frac{1}{\cos^2 x}}-3=0$$

$$so\ f(x)\ is\ increasing\ then\ f(x)\ge f(0)\ or\ \ 2sinx+tanx-3x\overset{f(0)=0}{\ge}0$$

$$or\ 2sinx+tanx\ge 3x\ \ proof\ complete.$$

$$9^{\sin a}+81^{\sin b}+729^{\sin c}+3^{\tan a}+9^{\tan b}+27^{\tan c}=$$

$$=3^{2sina}+3^{4sinb}+3^{6sinc}+3^{tana}+3^{2tanb}+3^{3tanc}=$$

$$=(3^{2sina}+3^{tana})+\left(3^{4sinb}+3^{2tanb}\right)+(3^{6sinc}+3^{3tanc})=$$

$$\overset{AM-GM}{\ge}2\times 3^{\frac{2sina+tana}{2}}+2\times 3^{\frac{2(2sinb+tanb)}{2}}+2\times 3^{\frac{3(2sinc+tanc)}{2}}\ge$$

$$\overset{(1)}{\ge}2\left(3^{\frac{3a}{2}}+3^{\frac{6b}{2}}+3^{\frac{9c}{2}}\right)\overset{AM-GM}{\ge}2\times 3\left(3^{\frac{3a+6b+9c}{2}}\right)^{\frac{1}{3}}=2\times 3\times 3^{\frac{a+2b+3c}{2}}=$$

$$=2\times 3^{\frac{a+2b+3c+2}{2}}=2\sqrt{3^{a+2b+3c+2}}$$

$$Equality\ holds\ for\ a=b=c=0.$$