ROMANIAN MATHEMATICAL MAGAZINE

If $x_1, x_2, \dots, x_n > 1$, $a, b, c > 0$ and $abc \ge a + b + c$, $n, m = 1, 2, \dots\dots$ then prove that:

$$\frac{a^2+b^2+c^2}{\left(\log_{x_1} x_2\right)^m}+\frac{a^4+b^4+c^4}{\left(\log_{x_2} x_3\right)^m}+\cdots+\frac{a^{2n}+b^{2n}+c^{2n}}{\left(\log_{x_n} x_1\right)^m} \ge 3n\sqrt{3^{n+1}}$$

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$$abc \ge a+b+c \text{ or } abc \overset{AM-GM}{\ge} 3\sqrt[3]{abc} \text{ or } (abc)^{\frac{2}{3}} \ge 3 \text{ or } abc \ge 3\sqrt{3} \ (1)$$

$$\frac{2}{3}+\frac{4}{3}+\cdots\frac{2n}{3}=\frac{2}{3}(1+2+\cdots+n)=\frac{2}{3}\cdot\frac{n(n+1)}{2}=\frac{n(n+1)}{3} \ (2)$$

$$\prod\left(\log_{x_1} x_2\right)^m=\left(\log_{x_1} x_2 \cdot \log_{x_2} x_3 \dots \log_{x_n} x_1\right)^m=$$

$$=\left(\frac{\log x_2}{\log x_1}\cdot\frac{\log x_3}{\log x_2}\cdots\frac{\log x_1}{\log x_n}\right)^m=(1)^m=1 \ (3)$$

$$a^2+b^2+c^2 \overset{AM-GM}{\ge} 3(abc)^{\frac{2}{3}}$$

$$a^4+b^4+c^4 \overset{AM-GM}{\ge} 3(abc)^{\frac{4}{3}} \dots, a^{2n}+b^{2n}+c^{2n} \overset{AM-GM}{\ge} 3(abc)^{\frac{2n}{3}}$$

$$\frac{a^2+b^2+c^2}{\left(\log_{x_1} x_2\right)^m}+\frac{a^4+b^4+c^4}{\left(\log_{x_2} x_3\right)^m}+\cdots+\frac{a^{2n}+b^{2n}+c^{2n}}{\left(\log_{x_n} x_1\right)^m} \ge$$

$$\ge \frac{3(abc)^{\frac{2}{3}}}{\left(\log_{x_1} x_2\right)^m}+\frac{3(abc)^{\frac{4}{3}}}{\left(\log_{x_2} x_3\right)^m}+\cdots+\frac{3(abc)^{\frac{2n}{3}}}{\left(\log_{x_n} x_1\right)^m} \ge$$

$$\overset{AM-GM}{\ge} 3n\sqrt[n]{\frac{(abc)^{\frac{2}{3}+\frac{4}{3}+\cdots\frac{2n}{3}}}{\prod\left(\log_{x_1} x_2\right)^m}} \overset{(2)\&(3)}{=} 3n\sqrt[n]{(abc)^{\frac{n(n+1)}{3}}} \overset{(1)}{\ge} 3n\sqrt[n]{\left(3\sqrt{3}\right)^{\frac{n(n+1)}{3}}}=3n\sqrt{3^{n+1}}$$

Equality holds for $x_1 = x_2 = \cdots = x_n$ & $a = b = c = \sqrt{3}$.