ROMANIAN MATHEMATICAL MAGAZINE

Prove that for all $x, y, z, w \in \mathbb{R}^+$ *the following inequality holds*:

$$\frac{xy+yz+zw}{x^2+y^2+z^2+w^2}+\frac{xyz+yzw}{x^3+y^3+z^3+w^3}+\frac{xyzw}{x^4+y^4+z^4+w^4}<\frac{2+\sqrt{5}}{4}+\frac{\sqrt[3]{4}}{3}$$

Proposed by Omkumar Rao-India

Solution by Jenish Rijal-Bardiya-Nepal

$$x^4+y^4+z^4+w^4 \overset{AM-GM}{\geq} 4\sqrt[4]{x^4y^4z^4w^4} \Leftrightarrow \frac{xyzw}{x^4+y^4+z^4+w^4} \overset{①}{\leq} \frac{1}{4}$$

$$y^3+z^3+x^3+w^3 \overset{Power\ Mean}{\geq} y^3+z^3+\frac{(x+w)^3}{4} \overset{AM-GM}{\geq} 3\sqrt[3]{y^3\cdot z^3\cdot\frac{(x+w)^3}{4}}=$$

$$=3\cdot\frac{xyz+yzw}{\sqrt[3]{4}} \Leftrightarrow \frac{xyz+yzw}{x^3+y^3+z^3+w^3} \overset{②}{\leq} \frac{\sqrt[3]{4}}{3}. \ \textit{Finally, Let} \quad J=\frac{1+\sqrt{5}}{4}$$

$$Jx^2+\frac{y^2}{4J} \overset{AM-GM}{\geq} 2\sqrt{Jx^2\cdot\frac{y^2}{4J}}=xy \qquad \textit{and} \qquad \frac{z^2}{4J}+Jw^2 \overset{AM-GM}{\geq} 2\sqrt{\frac{z^2}{4J}\cdot Jw^2}=zw$$

$$Jx^2-xy+\frac{y^2}{4J}+\frac{1}{2}(y-z)^2+\frac{z^2}{4J}-zw+Jw^2\geq 0$$

$$Jx^2+\frac{y^2}{4J}+\frac{y^2}{2}+\frac{z^2}{2}+\frac{z^2}{4J}+Jw^2\geq xy+yz+zw$$

$$\Leftrightarrow Jx^2+y^2\left(\frac{1}{4J}+\frac{1}{2}\right)+z^2\left(\frac{1}{2}+\frac{1}{4J}\right)+Jw^2\geq xy+yz+zw$$

$$\Leftrightarrow Jx^2+y^2\cdot J+z^2\cdot J+Jw^2\geq xy+yz+zw \Leftrightarrow J\left(x^2+y^2+z^2+w^2\right)\geq xy+yz+zw$$

$$\Leftrightarrow \frac{xy+yz+zw}{x^2+y^2+z^2+w^2} \overset{③}{\leq} J=\frac{1+\sqrt{5}}{4}$$

Summing ①, ② *and* ③ *we get*:

$$\frac{xy+yz+zw}{x^2+y^2+z^2+w^2}+\frac{xyz+yzw}{x^3+y^3+z^3+w^3}+\frac{xyzw}{x^4+y^4+z^4+w^4}<\frac{2+\sqrt{5}}{4}+\frac{\sqrt[3]{4}}{3}$$

Equality is not possible because ① *requires*

$x=y=z=w$ *for equality but that contradicts with* ③.