ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x^2 + y^2 + z^2 \le 3y$, then prove that :

$$\frac{1}{(x+1)^2} + \frac{4}{(y+2)^2} + \frac{8}{(z+3)^2} \ge 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{z^2}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} \overset{\text{Bergstrom}}{\ge} \frac{(z+1+1+1)^2}{4} \Rightarrow (z+3)^2 \le 4(z^2+3)$$

$$\therefore \frac{1}{(x+1)^2} + \frac{8}{(z+3)^2} \ge \frac{1}{2(x^2+1)} + \frac{8}{4(z^2+3)}$$

$$= \frac{1}{2(x^2+1)} + \frac{1}{z^2+3} + \frac{1}{z^2+3} \overset{\text{Bergstrom}}{\ge} \frac{9}{2(x^2+z^2)+8}$$

$$\overset{x^2+y^2+z^2 \le 3y}{\ge} \frac{9}{2(3y-y^2)+8} \Rightarrow \frac{1}{(x+1)^2} + \frac{4}{(y+2)^2} + \frac{8}{(z+3)^2} - 1$$

$$\ge \frac{9}{6y-2y^2+8} + \frac{4}{(y+2)^2} - 1$$

$$= \frac{9(y+2)^2 + 4(6y-2y^2+8) - (6y-2y^2+8)(y+2)^2}{(6y-2y^2+8)(y+2)^2} \overset{?}{\ge} 0$$

$$\Leftrightarrow 9(y+2)^2 + 4(6y-2y^2+8) - (6y-2y^2+8)(y+2)^2 \overset{?}{\ge} 0$$

$$\left(\because 6y-2y^2+8 = 2(3y-y^2)+8 \overset{x^2+y^2+z^2 \le 3y}{\ge} 2x^2+2z^2+8 > 0\right)$$

$$\Leftrightarrow 2y^4 + 2y^3 - 23y^2 + 4y + 36 \overset{?}{\ge} 0 \Leftrightarrow (y-2)^2(2y^2+10y+9) \overset{?}{\ge} 0 \to \text{true}$$

$$\Rightarrow \frac{1}{(x+1)^2} + \frac{4}{(y+2)^2} + \frac{8}{(z+3)^2} \ge 1 \ \forall\, x, y, z > 0 \,\middle|\, x^2+y^2+z^2 \le 3y,$$

$$'' ='' \text{ iff } x = z = 1 \text{ and } y = 2 \text{ (QED)}$$