ROMANIAN MATHEMATICAL MAGAZINE

If $a,\mathbf{b},\mathbf{c}>0$, *then* :

$$\frac{1}{a}+\frac{a}{b}+ab^2\geq\sqrt{3(1+a^2+b^2)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a}+\frac{a}{b}+ab^2\geq\sqrt{3(1+a^2+b^2)}\Leftrightarrow\left(\frac{b+a^2+a^2b^3}{ab}\right)^2\geq 3(1+a^2+b^2)$$

$$\Leftrightarrow a^4b^6+2a^4b^3+a^4+2a^2b+b^2\geq 3a^4b^2+a^2b^4+3a^2b^2\rightarrow(1)$$

Now, $a^4b^3+a^4b^3+a^4\overset{A-G}{\geq}3a^4b^2\Rightarrow 2a^4b^3+a^4\geq 3a^4b^2\rightarrow(\text{i})\therefore(\text{i})\Rightarrow$ in order to prove (1), it suffices to prove : $a^4b^5+2a^2+b\geq a^2b^3+3a^2b$

$$\Leftrightarrow a^4b^5+b+2a^2+a^2b^3\geq 2a^2b^3+3a^2b\rightarrow(2)$$

Now, $a^4b^5+b\overset{A-G}{\geq}2a^2b^3\rightarrow(\text{ii})\therefore(\text{ii})\Rightarrow$ to prove (2), it suffices to prove : $2a^2+a^2b^3\geq 3a^2b\Leftrightarrow b^3-3b+2\geq 0\Leftrightarrow(b+2)(b-1)^2\geq 0\rightarrow$ true $\because b>0$

$\Rightarrow(2)\Rightarrow(1)$ is true $\therefore\frac{1}{a}+\frac{a}{b}+ab^2\geq\sqrt{3(1+a^2+b^2)}\ \forall\, a,b,c>0$,

$''=''$ iff $a=b=c=1$ (QED)