

ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c > 0$ such that $bc + 2ca + 3ab \leq abc$, then prove that:

$$\frac{\sqrt{b^2 + 6ab + 4a^2}}{ab} + \frac{\sqrt{4c^2 + 18bc + 9b^2}}{bc} + \frac{\sqrt{9a^2 + 9ca + c^2}}{ca} \leq \sqrt{5}$$

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$$\text{Let } (x, y, z) = \left(\frac{1}{a}, \frac{2}{b}, \frac{3}{c}\right).$$

$$bc + 2ca + 3ab \leq abc \Leftrightarrow \frac{bc + 2ca + 3ab}{abc} \leq 1 \Leftrightarrow \frac{1}{a} + \frac{2}{b} + \frac{3}{c} \leq 1 \therefore x + y + z \leq 1$$

Now,

$$\begin{aligned} & \frac{\sqrt{b^2 + 6ab + 4a^2}}{ab} + \frac{\sqrt{4c^2 + 18bc + 9b^2}}{bc} + \frac{\sqrt{9a^2 + 9ca + c^2}}{ca} = \\ &= \sqrt{\frac{b^2}{a^2b^2} + \frac{6ab}{a^2b^2} + \frac{4a^2}{a^2b^2}} + \sqrt{\frac{4c^2}{b^2c^2} + \frac{18bc}{b^2c^2} + \frac{9b^2}{b^2c^2}} + \sqrt{\frac{9a^2}{c^2a^2} + \frac{9ca}{c^2a^2} + \frac{c^2}{c^2a^2}} = \\ &= \sqrt{\left(\frac{1}{a}\right)^2 + 3 \cdot \frac{1}{a} \cdot \frac{2}{b} + \left(\frac{2}{b}\right)^2} + \sqrt{\left(\frac{2}{b}\right)^2 + 3 \cdot \frac{2}{b} \cdot \frac{3}{c} + \left(\frac{3}{c}\right)^2} + \sqrt{\left(\frac{3}{c}\right)^2 + 3 \cdot \frac{3}{c} \cdot \frac{1}{a} + \left(\frac{1}{a}\right)^2} = \\ &= \sqrt{x^2 + 3xy + y^2} + \sqrt{y^2 + 3yz + z^2} + \sqrt{z^2 + 3zx + x^2} \end{aligned}$$

Thus, it suffices to prove that: $\sum_{cyc} \sqrt{x^2 + 3xy + y^2} \leq \sqrt{5}$

$$\text{Notice that: } x^2 + 3xy + y^2 = (x + y)^2 + xy \stackrel{AM-GM}{\leq} (x + y)^2 + \left(\frac{x + y}{2}\right)^2 = \frac{5}{4} \cdot (x + y)^2$$

$$\therefore \sum_{cyc} \sqrt{x^2 + 3xy + y^2} \leq \sum_{cyc} \sqrt{\frac{5}{4} \cdot (x + y)^2} = \sum_{cyc} \sqrt{5} \cdot \left(\frac{x + y}{2}\right) = \sqrt{5}(x + y + z) \leq \sqrt{5} \cdot 1 = \sqrt{5}$$

Equality holds if and only if $(a, b, c) = (3, 6, 9)$.