

ROMANIAN MATHEMATICAL MAGAZINE

If $a \geq 1, b > -1, \sqrt{ab + a - b - 1} + 1 \leq 2\sqrt{b + 1}$ then:

$$\frac{208\sqrt{ab + a - b - 1}}{25a + 27b + 2} + \frac{15(a - 1)}{b + 1} \leq 19$$

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$$ab + a - b - 1 = (a - 1)(b + 1)$$

Let $x = \sqrt{a - 1}, y = \sqrt{b + 1}$ then $x, y \geq 0$ as $a \geq 1$ & $b > -1$

Now, $\sqrt{ab + a - b - 1} + 1 \leq 2\sqrt{b + 1}$ or, $\sqrt{(a - 1)(b + 1)} + 1 \leq 2\sqrt{b + 1}$

$$xy + 1 \leq 2y \text{ or } x \leq 2 - \frac{1}{y}$$

$$x \leq y \left(\text{since } y - \left(2 - \frac{1}{y}\right) = \frac{(y - 1)^2}{y} \geq 0 \rightarrow y \geq 2 - \frac{1}{y} \right)$$

$$\text{Let } t = \frac{x}{y} \text{ \& } 0 \leq t \leq 1, a = x^2 + 1, b = y^2 - 1$$

$$25a + 27b + 2 = 25(x^2 + 1) + 27(y^2 - 1) + 2 = 25x^2 + 27y^2$$

$$\frac{208\sqrt{ab + a - b - 1}}{25a + 27b + 2} + \frac{15(a - 1)}{b + 1} = \frac{208xy}{25x^2 + 27y^2} + \frac{15x^2}{y^2} \stackrel{\text{as } x=ty}{=} \frac{208t}{25t^2 + 27} + 15t^2$$

$$19 - \left(\frac{208\sqrt{ab + a - b - 1}}{25a + 27b + 2} + \frac{15(a - 1)}{b + 1} \right) = 19 - \left(\frac{208t}{25t^2 + 27} + 15t^2 \right) =$$

$$\begin{aligned} &= \frac{19(25t^2 + 27) - 15t^2(25t^2 + 27) - 208t}{25t^2 + 27} = \\ &= \frac{-(t - 1)(375t^3 + 375t^2 + 305t + 513)}{25t^2 + 27} \geq 0 \text{ as } t \leq 1 \end{aligned}$$

Equality holds for $t = 1$ or, $x = y$ or, $a - 1 = b + 1$ or, $a = b + 2$ or, $a = 5, b = 3$