ROMANIAN MATHEMATICAL MAGAZINE

If $a > b > 0$ and $(a-b)\sqrt{ab} = 1$ then prove that :

$$\frac{1+4a^2b^2}{ab} + \frac{15}{a+b} \geq \frac{23}{2}$$

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$$(a-b)\sqrt{ab} = 1 \Rightarrow (a-b)^2 = \frac{1}{ab} \Rightarrow (a+b)^2 = 4ab + (a-b)^2$$

$$= 4ab + \frac{1}{ab} \Rightarrow a+b = \sqrt{\frac{4x^2+1}{x}}\ (x = ab) \text{ and so, } \frac{1+4a^2b^2}{ab} + \frac{15}{a+b} =$$

$$\frac{4x^2+1}{x} + \frac{15}{\sqrt{\frac{4x^2+1}{x}}} = t^2 + \frac{15}{t}\left(t = \sqrt{\frac{4x^2+1}{x}} \overset{\text{AM-GM}}{\geq} \sqrt{\frac{4x}{x}} = 2\right) \overset{?}{\geq} \frac{23}{2}$$

$$\Leftrightarrow 2t^3 - 23t + 30 \overset{?}{\geq} 0 \Leftrightarrow (t-2)\big((t-2)(2t+8)+1\big) \overset{?}{\geq} 0 \to \text{true} \because t \geq 2$$

$$\therefore \frac{1+4a^2b^2}{ab} + \frac{15}{a+b} \geq \frac{23}{2}; \text{equality occurs when } x = ab = \frac{1}{2} \text{ and then :}$$

$$a+b = \sqrt{\frac{4x^2+1}{x}} = 2 \therefore a(2-a) = \frac{1}{2} \Rightarrow 2a^2 - 4a + 1 = 0$$

$$\Rightarrow a = \frac{2 \pm \sqrt{2}}{2} \text{ and via } a+b = 2, b = \frac{2 \mp \sqrt{2}}{2} \text{ and since } a > b, \text{hence :}$$

$$a = \frac{2+\sqrt{2}}{2} \text{ and } b = \frac{2-\sqrt{2}}{2} \text{ and so, } \frac{1+4a^2b^2}{ab} + \frac{15}{a+b} \geq \frac{23}{2}$$

$$\forall a > b > 0 \,\big|\, (a-b)\sqrt{ab} = 1, '' = '' \text{ iff } \left\{\begin{matrix} a = \dfrac{2+\sqrt{2}}{2} \\ b = \dfrac{2-\sqrt{2}}{2} \end{matrix}\right\} \text{(QED)}$$