

ROMANIAN MATHEMATICAL MAGAZINE

If $-1 \leq x \leq 1$ then:

$$\frac{-4\sqrt{10}}{5} \leq 3x + \sqrt{1-x^2} \leq \sqrt{10}$$

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Let be $f: [-1, 1] \rightarrow \mathbb{R}; f(x) = 3x + \sqrt{1-x^2}, f'(x) = 3 - \frac{2x}{2\sqrt{1-x^2}} = 3 - \frac{x}{\sqrt{1-x^2}}$

$$f'(x) = 0 \Rightarrow 3 - \frac{x}{\sqrt{1-x^2}} = 0 \Leftrightarrow 3 = \frac{x}{\sqrt{1-x^2}}$$

$$\Leftrightarrow 9 = \frac{x^2}{1-x^2} \Leftrightarrow 9 - 9x^2 = x^2 \Leftrightarrow 10x^2 = 9 \Leftrightarrow x^2 = \frac{9}{10} \Leftrightarrow x = \frac{\pm 3\sqrt{10}}{10}$$

$$M = \max_{x \in [-1, 1]} f(x) = f\left(\frac{3\sqrt{10}}{10}\right) = 3 \cdot \frac{3\sqrt{10}}{10} + \sqrt{1 - \frac{9}{10}} = \frac{9\sqrt{10}}{10} + \frac{1}{\sqrt{10}} = \frac{9\sqrt{10}}{10} + \frac{\sqrt{10}}{10} = \sqrt{10}$$

$$\begin{aligned} m &= \min_{x \in [-1, 1]} f(x) = f\left(-\frac{3\sqrt{10}}{10}\right) = 3 \cdot \frac{-3\sqrt{10}}{10} + \sqrt{1 - \frac{9}{10}} = \\ &= \frac{-9\sqrt{10}}{10} + \frac{\sqrt{10}}{10} = \frac{-8\sqrt{10}}{10} = \frac{-4\sqrt{10}}{5} \Rightarrow -\frac{4\sqrt{10}}{5} \leq f(x) \leq \sqrt{10} \end{aligned}$$

$$\frac{-4\sqrt{10}}{5} \leq 3x + \sqrt{1-x^2} \leq \sqrt{10}$$