ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0$, $3x^2 + 4y^2 + 4xy - 9x - 10y + 8 = 0$ then:

$$\frac{7x + 11y + 13}{2x + 3y + 5} \geq 3$$

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$$We\ need\ to\ show: \frac{7x + 11y + 13}{2x + 3y + 5} \geq 3$$

$$7x + 11y + 13 \geq 6x + 9y + 15\ \ or, 2x + 2y \geq 2$$

$$Let\ L(x, y) = x + 2y$$

$$Let\ G(x, y) \equiv 3x^2 + 4y^2 + 4xy - 9x - 10y + 8$$

$$Extremum\ Values\ theorem(Weierstrass):$$

$$If\ a\ function\ f: X \to R\ is\ continuous\ on\ a\ compact\ set\ X\ (closed + bounded),$$

$$then\ f\ attains\ both\ maximum\ and\ minimum\ value\ on\ X.$$

$$X_{min}, X_{max} \in X\ such\ that\ f(X_{min}) \leq f(X) \leq f(X_{max})\quad (1)$$

$$3x^2 + 4y^2 + 4xy - 9x - 10y + 8 = 0$$

$$Comparing\ with\ the\ general\ equation\ of\ 2nd\ degree:$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\ we\ get:$$

$$a = 3, h = 2, b = 4, g = -\frac{9}{2}, f = -5, c = 8$$

$$\text{then } \Delta = \text{abc} + 2\text{fgh} - \text{af}^2 - \text{bg}^2 - \text{ch}^2 = -245 < 0 \text{ and}$$

$$\text{D} = \text{ab} - \text{h}^2 = 12 - 4 = 8 > 0$$

So the given equation represent an Elipse

$$\text{Set E} = \{(\text{x}, \text{y}) \in \text{R}^2 : 3x^2 + 4y^2 + 4xy - 9x - 10y + 8 = 0\}\ is\ \ a\ compact\ set\ (2)$$

$$Using\ Lagrange's\ multipliers\ at\ an\ extremum\ on\ the\ constraints\ there\ exist\ \lambda\ with$$

$$dL - \lambda dG = 0$$

$$or\ (dx + 2dy) - \lambda\big((6x + 4y - 9)dx + (8y + 4x - 10)dy\big) = 0$$

$$Comparing: 1 = \lambda(6x + 4y - 9)\ \ and\ 2 = \lambda(8y + 4x - 10)$$

$$Eliminating\ \lambda\ we\ get \frac{2}{1} = \frac{4x + 8\acute{y} - 10}{6x + 4y - 9}$$

$$or\ 12x + 8y - 18 = 4x + 8y - 10\ or\ \ 8x = 8\ or\ x = 1$$

$$Putting\ x = 1\ in\ G(x, y)\ we\ get\ 2y^2 - 3y + 1 = 0\ which\ implies\ y = 1, \frac{1}{2}$$

$$Now, L(x, y) = x + 2y\ then\ L(1, 1) = 3\ \ and\ L\left(1, \frac{1}{2}\right) = 2$$

$$\text{So using } (1) \text{ and } (2) \text{ we can say } 2 \le L(x,y) \le 3$$

$$2x + y \ge 2 \text{ or } \frac{7x + 11y + 13}{2x + 3y + 5} \ge 3$$

$$\text{Equality holds for } x = 1, y = \frac{1}{2}.$$