ROMANIAN MATHEMATICAL MAGAZINE

If $5+16.4^{x^2-2y} \le \left(5+16^{x^2-2y}\right).7^{2y-x^2+2}$, $\frac{x^2}{2} \ge y+1$ then:

$$\frac{3}{2} \le \frac{8x+4y+21}{2x+2y+5} \le 6$$

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$$Let\ t = x^2-2y\ then\ \frac{x^2}{2} \ge y+1 \Rightarrow x^2-2y \ge 2\ or, t \ge 2\ (1)$$

$$5+16.4^{x^2-2y} \le \left(5+16^{x^2-2y}\right).7^{2y-x^2+2}$$

$$5+16.4^{t} \le (5+16^{t}).7^{2-t}$$

$$5.7^{t-2}+16.4^t.7^{t-2} \le 5+16^t(\ multiply\ both\ sides\ by\ 7^{t-2}\ we\ get\ )$$

$$5.7^{t-2}+16.4^2 4^{t-2}.7^{t-2} \le 5+16^t$$

$$or\ 5.7^s+256.4^s.7^s \le 5+16^{s+2}\ \left(where\ s = t-2 \overset{(1)}{\ge} 0\right)$$

$$or\ \frac{5}{49}.7^s+\frac{256}{49}(28)^s \le \frac{5}{49}+\frac{256}{49}(16)^s(Dividing\ both\ sides\ by\ 49)$$

$$Clearly\ for\ s>0, 28^s>16^s, 7^s>1\ , so\ L.H.S\ strictly\ lager\ than\ R.H.S$$

$$for\ this\ the\ only\ way\ the\ inequality\ \ holds\ if\ s=0\ or$$

$$t = s+2 = 2\ or\ x^2-2y=2$$

$$\frac{8x+4y+21}{2x+2y+5} \overset{x^2-2y=2\ or\ y=\frac{x^2-2}{2}}{=} \frac{8x+4\left(\frac{x^2-2}{2}\right)+21}{2x+2.\frac{x^2-2}{2}+5} = \frac{2x^2+8x+17}{x^2+2x+3} = 2+\frac{4x+11}{x^2+2x+3}$$

$$We\ need\ to\ show:$$

$$\frac{8x+4y+21}{2x+2y+5} \ge \frac{3}{2}$$

$$or\ 2+\frac{4x+11}{x^2+2x+3} \ge \frac{3}{2}\ or\ 2(4x+11) \ge -x^2-2x-3\ or$$

$$x^2+10x+25 \ge 0\ or\ (x+5)^2 \ge 0\ true.\ Equality\ \ at\ x=-5, y=\frac{23}{2}$$

$$We\ need\ to\ show\ \frac{8x+4y+21}{2x+2y+5} \le 6\ or\ 2+\frac{4x+11}{x^2+2x+3} \le 6$$

$$or\ 4x+11 \le 4x^2+8x+12\ or\ 4x^2+4x+1 \ge 0\ or\ (2x+1)^2 \ge 0\ true$$

$$and\ equality\ at\ x=-\frac{1}{2}, y=\frac{1}{8}$$