ROMANIAN MATHEMATICAL MAGAZINE

If $0 < x, y, z < 1$ then:

$$\frac{4}{(\sqrt{x}+2\sqrt{y})(\sqrt{x}+2\sqrt{z})}+3(x-1)(y-1)(z-1)\geq\frac{4}{9}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ a=\sqrt{x}, b=\sqrt{y}, c=\sqrt{z}\ , clearly a, b, c\in(0,1)$$
$$(x-1)(y-1)(z-1)=(a^2-1)(b^2-1)(c^2-1)=-(1-a^2)(1-b^2)(1-c^2)\ (1)$$

$$(1-a^2)(1-b^2)(1-c^2)\overset{AM-GM}{\leq}\frac{(1-a^2+1-b^2+1-c^2)^3}{27}=$$
$$=\frac{\left(3-(a^2+b^2+c^2)\right)^3}{27}\leq$$
$$\overset{CBS}{\leq}\frac{\left(3-\frac{(a+b+c)^2}{3}\right)^3}{27}=\left(1-\frac{(a+b+c)^2}{9}\right)^3\overset{(a+b+c)^2=u}{=}\left(1-\frac{u}{9}\right)^3\ (2)$$

$$(\sqrt{x}+2\sqrt{y})(\sqrt{x}+2\sqrt{z})=(a+2b)(a+2c)\overset{AM-GM}{\leq}$$
$$\leq\frac{(a+2b+a+2c)^2}{4}=(a+b+c)^2\overset{(a+b+c)^2=u}{=}u\ (3)$$

$$We\ need\ to\ show:$$
$$\frac{4}{(\sqrt{x}+2\sqrt{y})(\sqrt{x}+2\sqrt{z})}+3(x-1)(y-1)(z-1)\geq\frac{4}{9}$$
$$or\ \frac{4}{u}-3\left(1-\frac{u}{9}\right)^3\geq\frac{4}{9}, clearly\ u=(a+b+c)^2\in(0,9)\ as\ a,b,c\in(0,1)$$

$$or\ \frac{4(9-u)(108-u(9-u)^2)}{243\ u}\geq 0\ or\ \frac{4(u-9)(u^3-18u^2+81u-108)}{243u}\geq 0$$

$$\frac{(u-3)^2(u-9)(u-12)}{243u}\geq 0\ as\ u\in(0,9)$$
$$Equalty\ holds\ for\ x=y=z=\frac{1}{3}.$$