ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0, 2a + b \leq 1$ then:

$$8a + \frac{1}{b} + \frac{1}{a} \geq 8$$

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$$2a + b \leq 1 \text{ or } b \leq 1 - 2a \text{ as } b > 0 \text{ so } 1 - 2a > 0 \text{ or } a < \frac{1}{2} \Rightarrow a \in \left(0, \frac{1}{2}\right)$$

$$8a + \frac{1}{b} + \frac{1}{a} \overset{b \leq 1-2a}{\geq} 8a + \frac{1}{a} + \frac{1}{1-2a} \text{ with } a \in \left(0, \frac{1}{2}\right)$$

$$\text{Let } f(a) = 8a + \frac{1}{a} + \frac{1}{1-2a}, f'(a) = 8 - \frac{1}{a^2} + \frac{2}{(1-2a)^2}, f''(a) = \frac{2}{a^3} + \frac{8}{(1-2a)^3}$$

$$\text{To get the critical point}, f'(a) = 0 \text{ gives } 8 - \frac{1}{a^2} + \frac{2}{(1-2a)^2} = 0$$

$$\text{or } 8a^2(1-2a)^2 - (1-2a)^2 + 2a^2 = 0 \text{ or } 32a^4 - 32a^3 + 6a^2 + 4a - 1 = 0$$

$$\text{or } (4a-1)(8a^3 - 6a^2 + 1) = 0 \text{ or } 4a - 1 = 0 \text{ gives } a = \frac{1}{4}$$

$$f''\left(\frac{1}{4}\right) = \frac{2}{\left(\frac{1}{4}\right)^3} + \frac{8}{\left(\frac{1}{2}\right)^3} > 0 \text{ so at } a = \frac{1}{4}, f(a) \text{ is min.}: f\left(\frac{1}{4}\right) = 8 \times \frac{1}{4} + 4 + 2 = 8$$

$$8a + \frac{1}{b} + \frac{1}{a} \overset{b \leq 1-2a}{\geq} 8a + \frac{1}{a} + \frac{1}{1-2a} = f(a) \geq 8$$

$$\text{Equality at } a = \frac{1}{4}, b = \frac{1}{2}.$$