ROMANIAN MATHEMATICAL MAGAZINE

If $x^3-6x^2+ax-b=0$ ***has 3 roots*** $x_1,x_2,x_3\geq 0$ ***prove that:***

$$8a-3b\leq 72$$

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$$x^3-6x^2+ax-b=0 \text{ then } \sum x_1=6, \sum x_1x_2=a, x_1x_2x_3=b$$

$$8a-3b=8\left(\sum x_1x_2\right)-3x_1x_2x_3$$

$$\text{Let } F(x_1,x_2,x_3)=8\left(\sum x_1x_2\right)-3x_1x_2x_3$$

F is symmetric, to find its Max value under the constraint $x_1+x_2+x_3=6$, we consider $x_1=x_2=t, x_3=6-2t$ and $t\in[0,3]$ as, $x_i\geq 0$

$$a=\sum x_1x_2=t^2+2(6-2t), b=x_1x_2x_3=t^2(6-2t)$$

$$F(t)=8\left(t^2+2(6-2t)\right)-3t^2(6-2t)=6t^3-42t^2+96t$$

$$F'(t)=18t^2-84t+96 \text{ to get critical point } F'(t)=0 \text{ gives}$$

$$18t^2-84t+96=0 \text{ or } (3t-8)(t-2)=0 \text{ or } t=\frac{8}{3},2$$

We find value of $F(t)$ a critical point and boundary $[0,3]$

$$F(0)=0, F(2)=6\times 2^3-42\times 2^2+96\times 2=72, F\left(\frac{8}{3}\right)=71.11, F(3)=72$$

so, Max value of $F(t)$ is 72 and $8a-3b\leq 72$ at $x_1=x_2=2, x_3=2$