ROMANIAN MATHEMATICAL MAGAZINE

$$\textit{If } a,b>0 \textit{ and the equation}$$
$$x^3-ax^2+bx-a=0 \textit{ has 3 roots } x_1,x_2,x_3\ge 1$$
$$\textit{then show that}:$$
$$a\ge\left(\frac{1}{4}+\frac{\sqrt{2}}{8}\right)(b+3)$$

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$$x^3-ax^2+bx-a=0 \textit{ has 3 roots } x_1,x_2,x_3\ge 1$$

$$\textit{then } x_1+x_2+x_3=a, x_1x_2+x_2x_3+x_3x_1=b,\quad x_1x_2x_3=a$$

$$\textit{According to the condition } x_1+x_2+x_3=x_1x_2x_3\ (1)$$

$$\frac{a}{b+3}=\frac{x_1+x_2+x_3}{x_1x_2+x_2x_3+x_3x_1+3} \textit{ now we are going to find } \text{Min} \textit{ value of } \frac{a}{b+3}$$

$$\textit{Since the function symmetric to get Min value we take}:$$
$$x_2=x_3=t\ge 1 \textit{ and } x_1=x$$
$$\textit{From } (1) \textit{ we get } x+2t=xt^2 \textit{ or } x=\frac{2t}{t^2-1}\ (2)$$
$$a=x_1+x_2+x_3=2t+x=2t+\frac{2t}{t^2-1}=\frac{2t^3}{t^2-1}$$
$$b=x_1x_2+x_2x_3+x_3x_1=t^2+2xt=t^2+2t.\frac{2t}{t^2-1}=t^2+\frac{4t^2}{t^2-1}=\frac{t^2(t^2+3)}{t^2-1}$$
$$\frac{a}{b+3}=\frac{\frac{2t^3}{t^2-1}}{\frac{t^2(t^2+3)}{t^2-1}+3}=\frac{2t^3}{t^4+6t^2-3}$$

$$\textit{Let } f(t)=\frac{2t^3}{t^4+6t^2-3}, f'(t)=\frac{-2t^2(t^2-3)^2}{(t^4+6t^2-3)^2}<0 \textit{ as } t\ge 1 \textit{ so } f(t) \textit{ is decreasing}$$
$$\textit{We declared } x=\frac{2t}{t^2-1} \textit{ so } t\ne 1, t>1 \textit{ now } x\ge 1\ (\textit{given})$$
$$\textit{For this } \frac{2t}{t^2-1}\ge 1 \textit{ or } t^2-2t-1\le 0 \textit{ or } \left(t-\left(\sqrt{2}+1\right)\right)\left(t-\left(1-\sqrt{2}\right)\right)\le 0$$
$$\textit{or } \left(1-\sqrt{2}\right)\le t\le\left(\sqrt{2}+1\right) \textit{ as } t>1 \textit{ then } 1<t<\left(\sqrt{2}+1\right)$$

$$\textit{Since, } f(t) \textit{ is decreasing so } f(t) \text{ min } \textit{at } t=\sqrt{2}+1$$

$$f(\sqrt{2}+1)=\frac{2(\sqrt{2}+1)^3}{(\sqrt{2}+1)^4+6(\sqrt{2}+1)^2-3}=\frac{2(2\sqrt{2}+7+3\sqrt{2})}{(3+2\sqrt{2})^2+6(3+2\sqrt{2})-3}$$
$$=\frac{2(5\sqrt{2}+7)}{32+24\sqrt{2}}=\frac{1}{4}\frac{(5\sqrt{2}+7)}{(3\sqrt{2}+4)}=\frac{1}{4}\frac{(3\sqrt{2}-4)(5\sqrt{2}+7)}{(3\sqrt{2}-4)(3\sqrt{2}+4)}=\frac{1}{4}\frac{2+\sqrt{2}}{2}=\left(\frac{1}{4}+\frac{\sqrt{2}}{8}\right)$$

$$\textit{Therefore}:$$

$$\frac{a}{b+3}\geq\left(\frac{1}{4}+\frac{\sqrt{2}}{8}\right)\ or\ a\geq\left(\frac{1}{4}+\frac{\sqrt{2}}{8}\right)(b+3)$$
$$when\ x_1=1,x_2=x_3=\sqrt{2}+1$$