ROMANIAN MATHEMATICAL MAGAZINE

$$\boldsymbol{If\ the\ equation\ \ x^5+ax^4+bx^3+cx^2+dx+1=0}$$
$$\boldsymbol{has\ 5\ real\ roots\ are\ different\ in\ pairs}$$
$$\boldsymbol{then\ prove\ that\ 2(a^2+d^2)>5(b+c)}$$

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$$x^5+ax^4+bx^3+cx^2+dx+1=0\quad(1)$$
$$Let\ r_1,r_2,r_3,r_4,r_5\ be\ the\ roots\ of\ the\ equation\ (1)$$

$$Relationships\ between\ roots\ and\ coefficient\ we\ get:$$
$$p_1=\sum_{i=1}^{5}r_i=-a$$
$$p_2=\sum_{1\le i<j\le 5}r_ir_j=b$$
$$p_3=\sum_{1\le i<j<k\le 5}r_ir_jr_k=-c$$
$$p_4=\sum_{1\le i<j<k<l\le 5}r_ir_jr_kr_l=d$$
$$p_5=\prod_{i=1}^{5}r_i=-1$$
$$\sum_{i=1}^{5}\frac{1}{r_i}=\left(\sum_{1\le i<j<k<l\le 5}r_ir_jr_kr_l\right)\times\frac{1}{\prod_{i=1}^{5}r_i}=\frac{p_4}{p_5}=-d\quad(1)$$
$$\left(\sum_{i=1}^{5}\frac{1}{r_i^2}\right)=\left(\sum_{i=1}^{5}\frac{1}{r_i}\right)^2-2\sum_{1\le i<j\le 5}\frac{1}{r_ir_j}=$$
$$=\left(\sum_{i=1}^{5}\frac{1}{r_i}\right)^2-2\left(\sum_{1\le i<j<k\le 5}r_ir_jr_k\right)\times\frac{1}{\prod_{i=1}^{5}r_i}\overset{(1)}{=}d^2-2\left(\frac{p_3}{p_5}\right)=d^2-2c\quad(2)$$

$$\left(\sum_{i=1}^{5}r_i^2\right)\left(\sum_{i=1}^{5}1^2\right)\overset{Cauchy-Schwarz}{>}\left(\sum_{i=1}^{5}r_i\right)^2$$
$$or\,5\left(\sum_{i=1}^{5}r_i^2\right)>(p_1)^2\ or\ 5\left(\left(\sum_{i=1}^{5}r_i\right)^2-2\sum_{1\le i<j\le 5}r_ir_j\right)>(p_1)^2$$

$$or\ 5\left(p_1^2-2p_2\right)>(p_1)^2\ \ or\ 5(a^2-2b)>a^2\ \ or\ 4a^2>10b\ or\ 2a^2>5b\ (3)$$

$$\left(\sum_{i=1}^{5}\frac{1}{r_i^2}\right)\left(\sum_{i=1}^{5}1^2\right)\overset{Cauchy-Schwarz}{>}\left(\sum_{i=1}^{5}\frac{1}{r_i}\right)^2\ or,5\left(\sum_{i=1}^{5}\frac{1}{r_i^2}\right)>\left(\sum_{i=1}^{5}\frac{1}{r_i}\right)^2$$

$$or\ 5(d^2-2c)\overset{(1)\&(2)}{>}d^2\ or,4d^2>10c\ \ or,2d^2>5c\ \ (4)$$

$$Adding\ (3)\ \&\ (4)we\ get:$$

$$2a^2+2d^2>5b+5c\ \ or,2(a^2+d^2)>5(b+c)$$