ROMANIAN MATHEMATICAL MAGAZINE

$$\textit{If the equation } ax^3+bx^2+cx+d=0,(a\neq 0)\textit{ has 3 roots}$$
$$x_1,x_2,x_3>0 \textit{ then prove that:}$$
$$x_1^7+x_2^7+x_3^7\geq -\frac{b^3c^2}{81a^5}$$

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Solution by Tapas Das-India

$$ax^3+bx^2+cx+d=0\ (1)$$

$$\textit{Relationship between roots and coefficient we get}$$
$$x_1+x_2+x_3=-\frac{b}{a}$$
$$x_1x_2+x_2x_3+x_3x_1=\frac{c}{a}$$
$$x_1x_2x_3=-\frac{d}{a}$$
$$x_1^7+x_2^7+x_3^7\overset{Chebyshev}{\geq}\frac{1}{3}\left(x_1^3+x_2^3+x_3^3\right)\left(x_1^4+x_2^4+x_3^4\right)\overset{CBS}{\geq}$$
$$\geq\frac{1}{3}\cdot\frac{(x_1+x_2+x_3)^3}{9}\cdot\frac{\left(x_1^2+x_2^2+x_3^2\right)^2}{3}\geq$$
$$\geq\frac{1}{81}(x_1+x_2+x_3)^3(x_1x_2+x_2x_3+x_3x_1)^2=\frac{1}{81}\left(-\frac{b}{a}\right)^3\left(\frac{c}{a}\right)^2=-\frac{b^3c^2}{81a^5}$$