ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b} > 0, n \geq 0$ and the equation : $x^3 - ax^2 + \mathrm{b}x - a = 0$ has 3 roots $x_1, x_2, x_3 > 1$, then prove that : $\frac{\mathrm{b}^{\mathrm{n}} - 3^{\mathrm{n}}}{a^{\mathrm{n}}} \geq 3^{\frac{\mathrm{n}}{2}} - \left(\frac{1}{\sqrt{3}}\right)^{\mathrm{n}}$

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Via Viete, $x_1 + x_2 + x_3 = a, x_1x_2 + x_2x_3 + x_3x_1 = \mathrm{b}$ and $x_1x_2x_3 = a$ and $\because (x_1 + x_2 + x_3)^2 \geq 3(x_1x_2 + x_2x_3 + x_3x_1)$ and

$$(x_1x_2 + x_2x_3 + x_3x_1)^2 \geq 3x_1x_2x_3(x_1 + x_2 + x_3) \therefore a^2 \geq 3\mathrm{b} \text{ and } \mathrm{b}^2 \overset{①}{\geq} 3a^2$$

$$\Rightarrow \mathrm{b}^2 \geq 3a^2 \geq 9\mathrm{b} \Rightarrow \mathrm{b} \overset{②}{\geq} 9 \therefore \frac{\mathrm{b}^{\mathrm{n}} - 3^{\mathrm{n}}}{a^{\mathrm{n}}} \overset{\text{via } ①}{\geq} \frac{\mathrm{b}^{\mathrm{n}} - 3^{\mathrm{n}}}{\left(\frac{\mathrm{b}}{\sqrt{3}}\right)^{\mathrm{n}}} (\because \mathrm{n} \geq 0) = 3^{\frac{\mathrm{n}}{2}} - \frac{3^{\mathrm{n}}.3^{\frac{\mathrm{n}}{2}}}{\mathrm{b}^{\mathrm{n}}} \overset{\text{via } ②}{\geq}$$

$$3^{\frac{\mathrm{n}}{2}} - \frac{3^{\mathrm{n}}.3^{\frac{\mathrm{n}}{2}}}{9^{\mathrm{n}}} (\because \mathrm{n} \geq 0) = 3^{\frac{\mathrm{n}}{2}} - \frac{1}{3^{\frac{\mathrm{n}}{2}}} \therefore \frac{\mathrm{b}^{\mathrm{n}} - 3^{\mathrm{n}}}{a^{\mathrm{n}}} \geq 3^{\frac{\mathrm{n}}{2}} - \left(\frac{1}{\sqrt{3}}\right)^{\mathrm{n}} \; \forall a, \mathrm{b} > 0, n \geq 0$$

such that the equation : $x^3 - ax^2 + \mathrm{b}x - a = 0$ has 3 roots $x_1, x_2, x_3 > 1$,

$'' ='' \text{ iff } \left(a = 3\sqrt{3}, \mathrm{b} = 9\right)$ (QED)