ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0, \mathrm{b}^2 + \mathrm{c}^2 = a^2$ and the equation : $-ax^2 + \mathrm{b}x + \mathrm{c} = 0$ has 2 real roots $x_1 < x_2$, then prove that : $-\sqrt{2} < x_1 < x_2 < \sqrt{2}$

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$$\text{Easy to see that : } x_1 = \frac{\mathrm{b} - \sqrt{\mathrm{b}^2 + 4a\mathrm{c}}}{2a} \text{ and } x_2 = \frac{\mathrm{b} + \sqrt{\mathrm{b}^2 + 4a\mathrm{c}}}{2a}$$

$$\therefore x_1 > \frac{-\sqrt{\mathrm{b}^2 + 4a\mathrm{c}}}{2a} \overset{?}{>} -\sqrt{2} \Leftrightarrow 2 \overset{?}{>} \frac{\mathrm{b}^2 + 4a\mathrm{c}}{4a^2} \overset{\mathrm{b}^2+\mathrm{c}^2 = a^2}{\Leftrightarrow} 8a^2 \overset{?}{>} a^2 - \mathrm{c}^2 + 4a\mathrm{c}$$

$$\Leftrightarrow (2a - \mathrm{c})^2 + 3a^2 \overset{?}{>} 0 \rightarrow \text{true} \therefore x_1 > -\sqrt{2} \text{ and again, } x_2 = \frac{\mathrm{b} + \sqrt{\mathrm{b}^2 + 4a\mathrm{c}}}{2a} \overset{?}{<} \sqrt{2}$$

$$\overset{\mathrm{b}^2+\mathrm{c}^2 = a^2}{\Leftrightarrow} \sqrt{\frac{a^2 - \mathrm{c}^2}{a^2}} + \sqrt{\frac{a^2 - \mathrm{c}^2 + 4a\mathrm{c}}{a^2}} \overset{?}{<} 2\sqrt{2}$$

$$\Leftrightarrow \left(\sqrt{1 - \mathrm{t}^2} + \sqrt{1 - \mathrm{t}^2 + 4\mathrm{t}}\right)^2 \overset{?}{<} \left(2\sqrt{2}\right)^2 \ \left(\mathrm{t} = \frac{\mathrm{c}}{a}\right)$$

$$\Leftrightarrow 2 - 2\mathrm{t}^2 + 4\mathrm{t} + 2.\sqrt{(1 - \mathrm{t}^2)(1 - \mathrm{t}^2 + 4\mathrm{t})} \overset{?}{<} 8$$

$$\Leftrightarrow \mathrm{t}^2 - 2\mathrm{t} + 3 \overset{?}{>} \sqrt{(1 - \mathrm{t}^2)(1 - \mathrm{t}^2 + 4\mathrm{t})} \Leftrightarrow (\mathrm{t}^2 - 2\mathrm{t} + 3)^2 \overset{?}{>} (1 - \mathrm{t}^2)(1 - \mathrm{t}^2 + 4\mathrm{t})$$

$$(\because \mathrm{t}^2 - 2\mathrm{t} + 3 = (\mathrm{t} - 1)^2 + 2 > 0) \Leftrightarrow 3\mathrm{t}^2 - 4\mathrm{t} + 2 \overset{?}{>} 0 \Leftrightarrow 2(\mathrm{t} - 1)^2 + \mathrm{t}^2 \overset{?}{>} 0$$

$$\therefore x_2 < \sqrt{2} \therefore -\sqrt{2} < x_1 < x_2 < \sqrt{2} \ \forall\, a, \mathrm{b}, \mathrm{c} > 0, \mathrm{b}^2 + \mathrm{c}^2 = a^2 \text{ such that}$$
$$\text{the equation : } -ax^2 + \mathrm{b}x + \mathrm{c} = 0 \text{ has 2 real roots } x_1 < x_2 \text{ (QED)}$$