If $x \in \mathbb{R}$ then:

$$2^{|\sin x|} + 2^{|\cos x|} \le 2 \cdot 2^{\frac{\sqrt{2}}{2}}$$

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$$We\ consider\ a\ function\ \phi(t) = \frac{2^t}{t}\ on\ (0,1]$$

$$\phi'(t) = \frac{2^t(t\log 2 - 1)}{t^2} < 0\ as\ \ t \in (0,1]\ so, \phi\ is\ \ strictly\ decreasing$$

$$which\ implies\ \phi\ \ is\ one - one\ function\ (Injective)$$

$$for\ this\ \phi\ (a) = \phi\ (b) \Rightarrow\ \ a = b\ (1)$$

$$let\ a = |\sin x|, b = |\cos x|\ then\ a^2 + b^2\ = 1\ (2)$$

$$Let\ L(a,b) = a^2 + b^2 + 1\ and\ g(a,b) = 2^a + 2^b$$

$$Using\ Lagranges\ multipliers\ \ \lambda\ \ we\ get\ dg = \lambda dL$$

$$or\ \ \log 2\left(2^a da + 2^b db\right) = \lambda\,(2a\,da + 2b\,db)$$

$$Comparing\ we\ get:$$

$$\log 2\ \ 2^a\ = \lambda 2a, \log 2\ \ 2^b\ = 2\lambda b$$

$$Combining\ we\ get:$$

$$\frac{\log 2\, 2^a}{2a} = \frac{\log 2\, 2^b}{2b} = \lambda$$

$$or \frac{2^a}{a} = \frac{2^b}{b}\ or\ a \overset{(1)}{=} b\ ,$$

$$from (2) we get\ a^2 + a^2\ = 1\ or, a = b = \frac{1}{\sqrt{2}}$$

$$Extremum\ should\ occurs\ \ for\ the\ function\ g(a,b) = 2^a + 2^b\ in\ [0,1]$$

$$when(\ a,b) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\ or\ in\ boundary\ (0,1),(1,0)$$

$$g\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2^{\frac{1}{\sqrt{2}}}\ + 2^{\frac{1}{\sqrt{2}}}\ = 2.2^{\frac{\sqrt{2}}{2}}$$

$$g(0,1) = 1 + 2 = 3\ and\ g(1,0) = 2 + 1 = 3$$

$$g(a,b) = 2^a + 2^b = 2^{|\sin x|} + 2^{|\cos x|} \le 2.2^{\frac{\sqrt{2}}{2}}\ \left(as\ 2.2^{\frac{\sqrt{2}}{2}} \approx 3.272 > 3\right)$$

$$Equality\ occurs\ for\ x = \frac{\pi}{4}$$