ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0,\ 3x^2 + 4y^2 + 4xy - 9x - 10y + 8 = 0$ then:

$$\frac{7x + 11y + 13}{2x + 3y + 5} \geq 3$$

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$$We\ need\ to\ show:$$

$$\frac{7x + 11y + 13}{2x + 3y + 5} \geq 3\ or\ 7x + 11y + 13 \geq 6x + 9y + 15\ \ or\ x + 2y \geq 2$$

$$Let\ L(x, y) = x + 2y$$

$$Let\ G(x, y) \equiv\ 3x^2 + 4y^2 + 4xy - 9x - 10y + 8 = 0$$

$$Extremum\ Values\ theorem(Weierstrass):$$

$$If\ a\ function\ f: X \to R\ is\ continuous\ on\ a\ compact\ set\ X\ (closed + bounded),$$

$$then\ f\ \ attains\ both\ maximum\ and\ minimum\ value\ on\ X.$$

$$X_{min}, X_{max} \in X\ such\ that\ f(\ X_{min}) \leq f(X) \leq f(X_{max})\quad (1)$$

$$3x^2 + 4y^2 + 4xy - 9x - 10y + 8 = 0$$

$$Comparing\ with\ general\ equation\ of\ 2nd\ degree$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\ we\ get\ a = 3, h = 2, b = 4,$$

$$g = -\frac{9}{2}, f = -5, c = 8$$

$$\text{then } \Delta = \text{abc} + 2\text{fgh} - \text{af}^2 - \text{bg}^2 - \text{ch}^2 = -74 < 0\ and$$

$$D = ab - \text{h}^2 = 12 - 4 = 8 > 0$$

$$\text{so, given equation represent Elipse . Clearly, set}$$

$$\text{E} = \{(\text{x}, \text{y}) \in \text{R}^2:\ 3x^2 + 4y^2 + 4xy - 9x - 10y + 8 = 0\} is\ \ a\ compact\ set\ (2)$$

$$Using\ Lagrange's\ multipliers\ at\ an\ extremum\ on\ the\ constraints\ there\ exist\ \lambda\ with$$

$$dL - \lambda dG = 0\ \ or, dL = \lambda dG$$

$$or\ (dx + 2dy) = \lambda\big((6x + 4y - 9)dx + (8y + 4x - 10)dy\big)$$

$$comparing\ , 1 = \lambda(6x + 4y - 9)\ or\ \frac{1}{6x + 4y - 9} = \lambda$$

$$and\ 2 = \lambda(8y + 4x - 10)\ or\ \frac{2}{8y + 4x - 10} = \lambda$$

$$therefore:$$

$$\frac{1}{6x + 4y - 9} = \frac{2}{8y + 4x - 10} or\ 2(6x + 4y - 9) = (8y + 4x - 10)\ or\ x = 1$$

$$Putting\ y = 1\ in\ G(x, y) = 0\ , we\ get:$$

$$3 + 4y^2 + 4y - 9 - 10y + 8 = 0\ or, 2y^2 - 3y + 1 = 0\ or$$

$$(y-1)(2y-1)=0 \text{ which implies } y=1,\frac{1}{2}$$

$$Now, L(x,y)=x+2y \text{ then } L(x=1,y=1)=3 \text{ and } L\left(x=1,y=\frac{1}{2}\right)=2$$

$$\text{so using } (1) \text{and } (2) \text{we can say } 2\le L(x,y)\le 3$$

$$L=x+2y\ge 2 \text{ or}, \frac{7x+11y+13}{2x+3y+5}\ge 3$$

$$\text{Equality holds for } x=1,y=\frac{1}{2}.$$