ROMANIAN MATHEMATICAL MAGAZINE

If $9x^2 - 3xy + y^2 = 2$ then:

$$\frac{2}{3} \le 9x^2 + 3xy + y^2 \le 6$$

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$$Let\ \ a = 3x, b = y\ then\ \ 9x^2 - 3xy + y^2 = a^2 - ab + b^2$$

$$and\ 9x^2 - 3xy + y^2 = 2\ or\ a^2 - ab + b^2 = 2\ or\ a^2 + b^2 = 2 + ab\ (1)$$

$$Let\ \ s = 9x^2 + 3xy + y^2 \overset{a=3x,b=y}{=} a^2 + ab + b^2 \overset{(1)}{=} 2 + ab + ab = 2(1 + ab)$$

$$We\ know\ that: (a^2 + b^2)^2 \overset{AM-GM}{\ge} 4a^2b^2$$
$$or\ (2 + ab)^2 \overset{(1)}{\ge} 4a^2b^2\ or\ \ 4 + 4ab + a^2b^2 \ge 4a^2b^2$$

$$or\ 3a^2b^2 - 4ab - 4 \le 0\ or\ (3ab + 2)(ab - 2) \le 0$$
$$or\ \ -\frac{2}{3} \le ab \le 2$$
$$or\ 1 - \frac{2}{3} \le 1 + ab \le 1 + 2\ or\ \frac{1}{3} \le (1 + ab) \le 3$$
$$or\ \ \frac{2}{3} \le 2(1 + ab) \le 6$$

$$s = 9x^2 + 3xy + y^2 \overset{a=3x,b=y}{=} a^2 + ab + b^2 \overset{(1)}{=} 2 + ab + ab = 2(1 + ab)$$
$$or\ \frac{2}{3} \le s \le 6\ \ or\ \ \frac{2}{3} \le 9x^2 + 3xy + y^2 \le 6$$