ROMANIAN MATHEMATICAL MAGAZINE

If $2x^2+4y^2-3xy=6$ then:

$$\frac{12\sqrt{2}}{4\sqrt{2}+3} \leq x^2+2y^2 \leq \frac{12\sqrt{2}}{4\sqrt{2}-3}$$

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$$\textit{Extremum Values theorem(Weierstrass):}$$
$$\textit{If a function } f: X \to R \textit{ is continuous on a compact set } X \textit{ (closed + bounded),}$$
$$\textit{then } f \textit{ attains both maximum and minimum values on } X.$$
$$X_{min}, X_{max} \in X \textit{ such that } f(X_{min}) \leq f(X) \leq f(X_{max}) \quad (1)$$

$$2x^2+4y^2-3xy=6 \textit{ or}, 2x^2+4y^2-3xy-6=0$$

$$\textit{comparing with general equation of 2nd degree}$$
$$ax^2+2hxy+by^2+2gx+2fy+c=0 \textit{ we get } a=2, h=-\frac{3}{2}, b=4, g=0,$$
$$f=0, c=-6$$
$$\text{then } \Delta = \text{abc}+2\text{fgh}-\text{af}^2-\text{bg}^2-\text{ch}^2=-\frac{246}{4}<0 \textit{ and}$$
$$D=ab-\text{h}^2=8-\frac{9}{4}=\frac{23}{4}>0$$
$$\text{so, given equation represents an elipse}$$
$$\text{clearly, set E} = \{(\text{x},\text{y}) \in \text{R}^2: 2x^2+4y^2-3xy-6=0\} \textit{ is a compact set} \ (2)$$

$$\textit{and to get the maximum and minimum value of given } f(x,y)=x^2+2y^2,$$
$$(x,y) \textit{ collect the values from the compact set } E$$
$$\textit{let}, y=kx \textit{ then we get from } 2x^2+4y^2-3xy-6=0 \textit{ as}$$
$$2x^2-3x(kx)+4(kx)^2=6$$
$$\textit{or } x^2(2-3k+4k^2)=6 \textit{ or}, x^2=\frac{6}{2-3k+4k^2} \quad (3)$$

$$\textit{Clearly denominator} > 0 \textit{ since } a=4>0 \textit{ and } D=(-3)^2-4.2.4==-23<0$$

$$\textit{Let } T = x^2+2y^2 \overset{y=kx}{=} x^2(1+2k^2) \overset{(3)}{=} \frac{6(1+2k^2)}{2-3k+4k^2}$$
$$\frac{dT}{dk}=\frac{(-6k^2+3)6}{(2-3k+4k^2)^2}, \textit{now } \frac{dT}{dk}=0 \textit{ gives } -6k^2+3=0 \textit{ or } k=\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$$

$$From\ (3) we\ get\ at\ k=\frac{1}{\sqrt{2}},\ x^2=\frac{6}{2-\frac{3}{\sqrt{2}}+\frac{4}{2}}=\frac{6\sqrt{2}}{4\sqrt{2}-3},\ y=kx=\frac{x}{\sqrt{2}}\ (4)$$

$$Similarly\ \ at\ k=-\frac{1}{\sqrt{2}},\ x^2=\frac{6\sqrt{2}}{4\sqrt{2}+3},\ y=-\frac{x}{\sqrt{2}}\ (5)$$

$$T=\ x^2+2y^2\overset{y=kx}{=}\ x^2(1+2k^2),\qquad T_{\left(at\ k=\frac{1}{\sqrt{2}}\right)}\overset{(4)}{=}\frac{6\sqrt{2}}{4\sqrt{2}-3}\left(1+2\times\frac{1}{2}\right)=\frac{12\sqrt{2}}{4\sqrt{2}-3}$$

$$T_{\left(at\ k=-\frac{1}{\sqrt{2}}\right)}\overset{(5)}{=}\frac{6\sqrt{2}}{4\sqrt{2}+3}\left(1+2\times\frac{1}{2}\right)=\frac{12\sqrt{2}}{4\sqrt{2}+3}$$

$$Now\ using\ theorem\ (1) we\ can\ say:$$

$$\frac{12\sqrt{2}}{4\sqrt{2}+3}\le x^2+2y^2\le\frac{12\sqrt{2}}{4\sqrt{2}-3}$$