ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}=\frac{3}{2}$ then prove that :

$$\frac{3}{x+y+z} \geq \frac{2}{xy+yz+zx}+\frac{1}{x^2+y^2+z^2}$$

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Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0$,

$b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b$

$\Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius

$$= s, R, r\ (say) \Rightarrow \sum_{cyc} x = s, \sum_{cyc} xy = 4Rr + r^2 \text{ and}$$

$$\sum_{cyc} x^2 = s^2 - 8Rr - 2r^2; \text{so } \frac{3}{x+y+z} \geq \frac{2}{xy+yz+zx}+\frac{1}{x^2+y^2+z^2}$$

$$\Leftrightarrow 2\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right).\frac{1}{x+y+z} \geq \frac{2}{xy+yz+zx}+\frac{1}{x^2+y^2+z^2}$$

$$\Leftrightarrow \frac{2}{s}.\sum_{cyc}\frac{1}{a} \geq \frac{2}{4Rr+r^2}+\frac{1}{s^2-8Rr-2r^2}$$

$$\Leftrightarrow \frac{2}{s}.\frac{s^2+4Rr+r^2}{4Rrs} \geq \frac{2s^2-12Rr-3r^2}{(4Rr+r^2)(s^2-8Rr-2r^2)} \Leftrightarrow 2r^4(44R-7r)(R-2r) \geq 0$$

$$\rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \therefore \frac{3}{x+y+z} \geq \frac{2}{xy+yz+zx}+\frac{1}{x^2+y^2+z^2}$$

$$\forall\, x, y, z > 0 \,\Big|\, \frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}=\frac{3}{2}, ''='' \text{ iff } x=y=z=1 \text{ (QED)}$$