ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0$, $x^2 + y^2 + xy - 3x - 3y + 2 = 0$ then:

$$\frac{3x + 2y + 1}{x + y + 6} \leq 1$$

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$$We\ need\ to\ show:$$

$$\frac{3x + 2y + 1}{x + y + 6} \leq 1\ or, 3x + 2y + 1 \leq x + y + 6\ \ or, 2x + y \leq 5$$

$Let\ L(x, y) = 2x + y\ .\ Let\ G(x, y) \equiv x^2 + y^2 + xy - 3x - 3y + 2 = 0$

$$Extremum\ Value\ theorem(Weierstrass):$$

$$If\ a\ function\ f: X \to R\ is\ continuous\ on\ a\ compact\ set\ X\ (closed + bounded),$$

$$then\ attains\ both\ maximum\ and\ minimum\ value\ on\ X.$$

$$X_{min}, X_{max} \in X\ such\ that\ f(\ X_{min}) \leq f(X) \leq f(X_{max}) \quad (1)$$

$$x^2 + y^2 + xy - 3x - 3y + 2 = 0$$

$comparing\ with\ general\ equation\ of\ 2nd\ degree:$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\ we\ get:$$

$$a = 1, h = \frac{1}{2}, b = 1, g = -\frac{3}{2}, f = -\frac{3}{2}, c = 2$$

$$\text{then } \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = -\frac{3}{4} < 0 \text{ and } D = ab - h^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0$$

so the given equation represent an elipse

$$\text{clearly, set } E = \{(x, y) \in R^2: x^2 + y^2 + xy - 3x - 3y + 2 = 0\ \}is\ \ a\ compact\ set\ (2)$$

$Using\ Lagrange's\ multipliers\ at\ an\ extremum\ on\ the\ constraints\ there\ exist\ \lambda\ with$

$$dL - \lambda dG = 0$$

$$(2dx + dy) - \lambda\big((2x + y - 3)dx + (2y + x - 3)dy\big) = 0$$

$$(2dx + dy) = \lambda\big((2x + y - 3)dx + (2y + x - 3)dy\big)$$

$$comparing:\ 2 = \lambda(2x + y - 3)\ or\ \frac{2}{(2x + y - 3)} = \lambda$$

$$and\ 1 = \lambda(2y + x - 3)\ or\ \frac{1}{(2y + x - 3)} = \lambda$$

$$Therefore\ \frac{2}{(2x + y - 3)} = \frac{1}{(2y + x - 3)}$$

$$2(2y + x - 3) = (2x + y - 3)\ or\ y = 1$$

$$putting\ y = 1\ in\ G(x, y) we\ get\ x^2 - 2x = 0\ which\ implies\ x = 0, 2$$

$$Now, L(x,y) = 2x + y \ then \ L(0,1) = 1 \ and \ L(2,1) = 5$$

$$so \ using \ (1) and \ (2) we \ can \ say \ 1 \leq L(x,y) \leq 5$$

$$2x + y \leq 5 \ or, \frac{3x + 2y + 1}{x + y + 6} \leq 1 \ ,$$

$$Equaity \ holds \ for \ \ x = 2, y = 1$$