ROMANIAN MATHEMATICAL MAGAZINE

If $x^2+4y^2-2x+8y+1=0$ then:

$$-10\le\frac{-2x+10y-14}{x+2y+1}\le 1$$

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$$\textit{We know that for any real number } A \textit{ and } B:$$
$$-\sqrt{A^2+B^2}\le A\cos\theta+B\sin\theta\le\sqrt{A^2+B^2}\ \ (1)$$
$$x^2+4y^2-2x-8y+1=0\ or,(x^2-2x+1)+4(y^2-2y+1)-4=0\ or,$$
$$or,(x-1)^2+4(y-1)^2=4\ (2)$$
$$from\ (2) we\ take\ x-1=2\ \cos\theta\ \ or,x=1+2\cos\theta\ \ ,y-1=\sin\theta\ \ or,y=1+\sin\theta$$
$$let\ t=\frac{-2x+10y-14}{x+2y+1}\overset{x=1+2\cos\theta,y=1+\sin\theta}{=}\frac{-2(1+2\cos\theta)+10(1+\sin\theta)-14}{1+2\cos\theta+2(1+\sin\theta)+1}$$
$$=\frac{-3-2\cos\theta+5\sin\theta}{2+\cos\theta+\sin\theta}$$
$$now\ t=\frac{-3-2\cos\theta+5\sin\theta}{2+\cos\theta+\sin\theta}\ \ or,t(2+\cos\theta+\sin\theta)=-3-2\cos\theta+5\sin\theta$$
$$or,(-2-t)\cos\theta+(5-t)\sin\theta=2t+3$$
$$or,|2t+3|=|\,(-2-t)\cos\theta+(5-t)\sin\theta|\overset{(1)}{\le}\sqrt{(2+t)^2+(5-t)^2}$$
$$or,(2t+3)^2\le(t+2)^2+(t-5)^2$$
$$or,4t^2+12t+9\le\ t^2+4t+4+t^2-10t+25\ or,2t^2+18t-20\le 0$$
$$or,t^2+9t-10\le 0\ \ or,(t+10)(t-1)\le 0\ \ or,-10\le t\le 1$$
$$or,-10\le\frac{-2x+10y-14}{x+2y+1}\le 1$$
$$Equality\ occurs\ when\ t=\frac{-2x+10y-14}{x+2y+1}=1\ at\ x=-\frac{1}{5},y=\frac{9}{5}$$
$$Equality\ occurs\ when\ t=\frac{-2x+10y-14}{x+2y+1}=-10\ at\ x=\frac{1}{17},y=\frac{2}{17}$$