ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ and $a + \mathrm{b} + \mathrm{c} = 3$, then prove that :

$$\frac{25a^2}{\sqrt{2a^2+16a\mathrm{b}+7\mathrm{b}^2}}+\frac{25\mathrm{b}^2}{\sqrt{2\mathrm{b}^2+16\mathrm{bc}+7\mathrm{c}^2}}+\frac{\mathrm{c}^2(3+a)}{a}\geq 14$$

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$$\sqrt{2a^2+16a\mathrm{b}+7\mathrm{b}^2}\overset{?}{\leq}3\mathrm{b}+2a\Leftrightarrow 9\mathrm{b}^2+4a^2+12a\mathrm{b}\overset{?}{\geq}2a^2+16a\mathrm{b}+7\mathrm{b}^2$$

$$\Leftrightarrow 2(a-\mathrm{b})^2\overset{?}{\geq}0\rightarrow\text{true}\therefore\sqrt{2a^2+16a\mathrm{b}+7\mathrm{b}^2}\overset{①}{\leq}3\mathrm{b}+2a\text{ and also,}$$

$$\sqrt{2\mathrm{b}^2+16\mathrm{bc}+7\mathrm{c}^2}\overset{?}{\leq}3\mathrm{c}+2\mathrm{b}\Leftrightarrow 9\mathrm{c}^2+4\mathrm{b}^2+12\mathrm{bc}\overset{?}{\geq}2\mathrm{b}^2+16\mathrm{bc}+7\mathrm{c}^2$$

$$\Leftrightarrow 2(\mathrm{b}-\mathrm{c})^2\overset{?}{\geq}0\rightarrow\text{true}\therefore\sqrt{2\mathrm{b}^2+16\mathrm{bc}+7\mathrm{c}^2}\overset{②}{\leq}3\mathrm{c}+2\mathrm{b}\text{ and so, ① and ② }\Rightarrow$$

$$\frac{25a^2}{\sqrt{2a^2+16a\mathrm{b}+7\mathrm{b}^2}}+\frac{25\mathrm{b}^2}{\sqrt{2\mathrm{b}^2+16\mathrm{bc}+7\mathrm{c}^2}}+\frac{\mathrm{c}^2(3+a)}{a}\geq$$

$$\geq\frac{25a^2}{3\mathrm{b}+2a}+\frac{25\mathrm{b}^2}{3\mathrm{c}+2\mathrm{b}}+\frac{9\mathrm{c}^2}{3a}+\mathrm{c}^2\overset{\text{Bergstrom}}{\geq}\frac{(5a+5\mathrm{b}+3\mathrm{c})^2}{5\mathrm{b}+5a+3\mathrm{c}}+\mathrm{c}^2\overset{a+\mathrm{b}+\mathrm{c}=3}{=}$$

$$5(3-\mathrm{c})+3\mathrm{c}+\mathrm{c}^2=14+\mathrm{c}^2-2\mathrm{c}+1=14+(\mathrm{c}-1)^2\geq 14$$

$$\therefore\frac{25a^2}{\sqrt{2a^2+16a\mathrm{b}+7\mathrm{b}^2}}+\frac{25\mathrm{b}^2}{\sqrt{2\mathrm{b}^2+16\mathrm{bc}+7\mathrm{c}^2}}+\frac{\mathrm{c}^2(3+a)}{a}\geq 14$$

$$\forall\, a,\mathrm{b},\mathrm{c}>0\,\middle|\,a+\mathrm{b}+\mathrm{c}=3,''=''\text{ iff }a=\mathrm{b}=\mathrm{c}=1\text{ (QED)}$$