ROMANIAN MATHEMATICAL MAGAZINE

If $a \geq b \geq c > 0$ then prove that :

$$\frac{a^3b}{c} + b^2c + c^3 \geq \frac{a^2b^2}{c} + bc^2 + c^2a$$

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$$\frac{a^3b}{c} + b^2c + c^3 \geq \frac{a^2b^2}{c} + bc^2 + c^2a$$

$$\Leftrightarrow a^3b + b^2c^2 + c^4 - a^2b^2 - bc^3 - c^3a \geq 0$$

$$\Leftrightarrow (b^2c^2 - a^2b^2) - (bc^3 - a^3b) + (c^4 - c^3a) \geq 0$$

$$\Leftrightarrow (c-a)(b^2c + ab^2 - bc^2 - abc - a^2b + c^3) \geq 0$$

$$\Leftrightarrow (c-a)\Big((c^3 - bc^2) + (ab^2 - a^2b) + (b^2c - abc)\Big)$$

$$\Leftrightarrow (c-a)\Big(c^2(c-b) + ab(b-a) + bc(b-a)\Big) \geq 0$$

$$\Leftrightarrow (a-c)\Big(c^2(b-c) + ab(a-b) + bc(a-b)\Big) \geq 0 \rightarrow \text{true} \because a \geq b \geq c > 0$$

$$\therefore \frac{a^3b}{c} + b^2c + c^3 \geq \frac{a^2b^2}{c} + bc^2 + c^2a \ \forall\ a \geq b \geq c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$