ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0$ and $(\sqrt{x}+1)(2\sqrt{y}+4)+y \ge 13$ then prove that :

$$\frac{x^4}{y}+\frac{y^3}{x}+y \ge 3$$

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$$(\sqrt{x}+1)(2\sqrt{y}+4)+y \ge 13 \Rightarrow 2\sqrt{xy}+4\sqrt{x}+2\sqrt{y}+y \ge 9$$

$$\Rightarrow (2\sqrt{x}+\sqrt{y})(2+\sqrt{y}) \ge 9 \Rightarrow 9 \le (2\sqrt{x}+\sqrt{y})(2+\sqrt{y}) \overset{\text{CBS}}{\le}$$

$$\sqrt{3}.\sqrt{2x+y}.\sqrt{3}.\sqrt{y+2} \overset{\text{AM-GM}}{\le} \frac{3}{2}(2x+2y+2) \Rightarrow x+y \overset{①}{\ge} 2$$

$\boxed{\text{Case 1}}\; x \le \frac{1}{4}$ and then : $y \overset{\text{via ①}}{\ge} 2-x \ge 2-\frac{1}{4}=\frac{7}{4}$ and so, $\frac{x^4}{y}+\frac{y^3}{x}+y-3$

$$> \frac{y^3}{x}+y-3 \ge 4y^3+y-3 > 4\left(\frac{7}{4}\right)^3+\frac{7}{4}-3 > 0 \therefore \frac{x^4}{y}+\frac{y^3}{x}+y > 3$$

$\boxed{\text{Case 2}}\; \frac{1}{4} < x < 2$ and then : $\frac{x^4}{y}+\frac{y^3}{x}-(3-y) = \frac{x^4}{y}+\frac{y^4}{xy}-(3-y) \overset{\text{Bergstrom}}{\ge}$

$$\frac{(x^2+y^2)^2}{y(x+1)}-(3-y) \overset{\substack{\text{AM-GM} \\ \text{and} \\ \text{via ①}}}{\ge} \frac{(x^2+(2-x)^2).2xy}{y(x+1)}-(x+1) =$$

$$\frac{2x(x^2+(2-x)^2)-(x+1)^2}{x+1} = \frac{(x-1)^2(4x-1)}{x+1} \ge 0 \because \frac{1}{4} < x \therefore \frac{x^4}{y}+\frac{y^3}{x}+y \ge 3$$

$\boxed{\text{Case 3}}\; x \ge 2$ and then : $\frac{x^4}{y}+\frac{y^3}{x}-(3-y) = \frac{x^4}{y}+\frac{y^4}{xy}-(3-y) \overset{\text{Bergstrom}}{\ge}$

$$\frac{(x^2+y^2)^2}{y(x+1)}-(3-y) \overset{\substack{\text{AM-GM} \\ \text{and} \\ \text{via ①}}}{\ge} \frac{2xy(x^2+y^2)}{y(x+1)}-(x+1) > \frac{2x(x^2)}{x+1}-(x+1)$$

$$= \frac{2x^3-x^2-2x-1}{x+1} = \frac{(x-2)(2x^2+3x+4)+7}{x+1} > 0 \because x \ge 2$$

$$\therefore \frac{x^4}{y}+\frac{y^3}{x}+y > 3 \text{ and so, combining all cases, } \frac{x^4}{y}+\frac{y^3}{x}+y \ge 3\; \forall$$

$$x, y > 0 \;\middle|\; (\sqrt{x}+1)(2\sqrt{y}+4)+y \ge 13, ''='' \text{ iff } x=y=1 \text{ (QED)}$$