ROMANIAN MATHEMATICAL MAGAZINE

If $xz+yz+2y=1$ and $2xz+yz+x+3y=2$

then prove that : $0\le xy(1+z)\le\frac{1}{4}$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(2xz+yz+x+3y)-(xz+yz+2y)=2-1\Rightarrow xz+x+y=1$$
$$\Rightarrow x(1+z)=1-y\Rightarrow xy(1+z)=y(1-y)=y-y^2-\frac{1}{4}+\frac{1}{4}=\frac{1}{4}-\frac{4y^2-4y+1}{4}$$
$$=\frac{1}{4}-\frac{(2y-1)^2}{4}\le\frac{1}{4}$$

Now, $2(xz+yz+2y)=2xz+yz+x+3y\Rightarrow y+yz=x\Rightarrow y(1+z)=x$

$\Rightarrow xy(1+z)=x^2\ge 0$ and so, $0\le xy(1+z)\le\frac{1}{4}$ whenever $xz+yz+2y=1$

and $2xz+yz+x+3y=2$, " = " for LH inequality iff $x=0$

and " = " for RH inequality iff $y=\frac{1}{2}$ (QED)