ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ and $(a+\mathrm{b})^2 + 2\mathrm{c}^2 \geq 2$ then prove that :

$$5(a^2+\mathrm{b}^2+\mathrm{c}^2) - \left(a+\mathrm{b}+\sqrt{2}\mathrm{c}\right)^2 - \sqrt{\frac{(a+\mathrm{b})^2}{2} + \mathrm{c}^2} \geq 0$$

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$$5(a^2+\mathrm{b}^2+\mathrm{c}^2) - \left(a+\mathrm{b}+\sqrt{2}\mathrm{c}\right)^2 - \sqrt{\frac{(a+\mathrm{b})^2}{2} + \mathrm{c}^2} \overset{\mathrm{CBS}}{\geq}$$

$$5\left(\frac{(a+\mathrm{b})^2}{2} + \mathrm{c}^2\right) - 2((a+\mathrm{b})^2 + 2\mathrm{c}^2) - \sqrt{\frac{(a+\mathrm{b})^2 + 2\mathrm{c}^2}{2}} = \frac{5x}{2} - 2x - \sqrt{\frac{x}{2}}$$

$$(x = (a+\mathrm{b})^2 + 2\mathrm{c}^2) = \sqrt{\frac{x}{2}}.\left(\sqrt{\frac{x}{2}} - 1\right) = \sqrt{\frac{x}{2}}.\frac{\frac{x}{2} - 1}{\sqrt{\frac{x}{2}} + 1} \geq 0 \because x \geq 2 \Rightarrow \frac{x}{2} - 1 \geq 0$$

$$\therefore 5(a^2+\mathrm{b}^2+\mathrm{c}^2) - \left(a+\mathrm{b}+\sqrt{2}\mathrm{c}\right)^2 - \sqrt{\frac{(a+\mathrm{b})^2}{2} + \mathrm{c}^2} \geq 0 \ \forall\, a, \mathrm{b}, \mathrm{c} > 0$$

with $(a+\mathrm{b})^2 + 2\mathrm{c}^2 \geq 2, '' ='' \text{ iff } a = \mathrm{b} \ \wedge\ a+\mathrm{b} = \sqrt{2}\mathrm{c} \ \wedge\ (a+\mathrm{b})^2 + 2\mathrm{c}^2 = 2$

$$\Rightarrow '' ='' \text{ iff } \left(a = \mathrm{b} = \frac{1}{2}, \mathrm{c} = \frac{1}{\sqrt{2}}\right) \text{ (QED)}$$