$$\text{If } x+z-yz=1 \text{ and } y-3z+xz=1, \text{then prove that :}$$

$$6-2\sqrt{5}\le x^2+y^2\le 6+2\sqrt{5}$$

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$$\text{Easy to see that if : } z=0, \text{then : } x=y=1 \text{ and then :}$$
$$6-2\sqrt{5}<x^2+y^2=2<6+2\sqrt{5} \text{ and we now focus on : } z\ne 0 \text{ and then :}$$
$$x\ne 1, y\ne 1 \text{ and } x+z-yz=1 \Rightarrow z(1-y)\overset{(1)}{=}1-x \text{ and also}, y-3z+xz=1$$
$$\Rightarrow z(x-3)\overset{(2)}{=}1-y \text{ and } (1)\div(2)\Rightarrow \frac{1-y}{x-3}=\frac{1-x}{1-y}\Rightarrow 4x-x^2-3=y^2-2y+1$$
$$\Rightarrow x^2+y^2=2(2x+y)-4\overset{\text{CBS}}{\le}2\sqrt{5(x^2+y^2)}-4$$
$$\Rightarrow t^2-2\sqrt{5}.t+4\overset{(*)}{\le}0\ \left(t=\sqrt{x^2+y^2}\right)$$
$$\text{Now}, (*)\Rightarrow t\le\frac{2\sqrt{5}+\sqrt{20-16}}{2}\Rightarrow\sqrt{x^2+y^2}\le\sqrt{5}+1$$
$$\Rightarrow x^2+y^2\le 6+2\sqrt{5}\rightarrow\text{(i)}$$
$$\text{Also}, (*)\Rightarrow t\ge\frac{2\sqrt{5}-\sqrt{20-16}}{2}\Rightarrow\sqrt{x^2+y^2}\ge\sqrt{5}-1\Rightarrow x^2+y^2\ge 6-2\sqrt{5}$$
$$\rightarrow\text{(ii)} \therefore \text{(i) and (ii)}\Rightarrow 6-2\sqrt{5}\le x^2+y^2\le 6+2\sqrt{5}, \text{equalities occur when :}$$
$$x=2y\Rightarrow \text{when : } 5y^2-10y+4=0\Rightarrow \text{when : } y=\frac{5\pm\sqrt{5}}{5} \text{ and so,}$$
$$\text{equality for : } 6-2\sqrt{5}\le x^2+y^2 \text{ occurs when : }\begin{pmatrix} x=\dfrac{10-2\sqrt{5}}{5} \\ y=\dfrac{5-\sqrt{5}}{5}\end{pmatrix} \text{ and}$$
$$\text{equality for : } x^2+y^2\le 6+2\sqrt{5} \text{ occurs when : }\begin{pmatrix} x=\dfrac{10+2\sqrt{5}}{5} \\ y=\dfrac{5+\sqrt{5}}{5}\end{pmatrix} \text{(QED)}$$