ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } x \in \left[0, \frac{\pi}{2}\right], \text{then prove that}:$$

$$\sin x \le \frac{4}{\pi}x - \frac{4}{\pi^2}x^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } f(x) = \frac{\sin x}{x} + \frac{4x}{\pi^2} - \frac{4}{\pi}\ \forall\, x \in \left(0, \frac{\pi}{2}\right] \text{ and then}:$$

$$f'(x) = -\frac{\sin x}{x^2} + \frac{\cos x}{x} + \frac{4}{\pi^2} \text{ and } f''(x) = -\frac{(x^2-2)\sin x + 2x\cos x}{x^3}$$

$$\text{and now, let } F(x) = (x^2-2)\sin x + 2x\cos x\ \ \forall\, x \in \left[0, \frac{\pi}{2}\right] \text{ and then}:$$

$$F'(x) = x^2\cos x \ge 0\ \forall\, x \in \left[0, \frac{\pi}{2}\right] \Rightarrow F(x) \text{ is} \uparrow \text{on } \left[0, \frac{\pi}{2}\right] \Rightarrow F(x) \ge F(0) = 0,$$

$$'' ='' \text{ iff } x = 0 \Rightarrow (x^2-2)\sin x + 2x\cos x > 0\ \forall\, x \in \left(0, \frac{\pi}{2}\right]$$

$$\Rightarrow f''(x) < 0\ \forall\, x \in \left(0, \frac{\pi}{2}\right] \Rightarrow f'(x) \text{ is} \downarrow \text{on } \left(0, \frac{\pi}{2}\right] \Rightarrow f'(x) \ge f'\left(\frac{\pi}{2}\right)$$

$$= -\frac{4}{\pi^2} + 0 + \frac{4}{\pi^2} = 0 \Rightarrow f'(x) \ge 0\ \forall\, x \in \left(0, \frac{\pi}{2}\right] \Rightarrow f(x) \text{ is} \uparrow \text{on } \left(0, \frac{\pi}{2}\right]$$

$$\Rightarrow f(x) \le f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} + \frac{4}{\pi^2}\cdot\frac{\pi}{2} - \frac{4}{\pi} = 0 \Rightarrow \frac{\sin x}{x} + \frac{4x}{\pi^2} - \frac{4}{\pi} \le 0\ \forall\, x \in \left(0, \frac{\pi}{2}\right]$$

$$\Rightarrow \sin x \le \frac{4}{\pi}x - \frac{4}{\pi^2}x^2\ \forall\, x \in \left(0, \frac{\pi}{2}\right] \text{ and } \because \sin(0) = \frac{4}{\pi}(0) - \frac{4}{\pi^2}(0)^2$$

$$\therefore \sin x \le \frac{4}{\pi}x - \frac{4}{\pi^2}x^2\ \forall\, x \in \left[0, \frac{\pi}{2}\right], '' ='' \text{ iff } x = 0 \text{ or } x = \frac{\pi}{2}\ \text{(QED)}$$