ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x^3 + y^2 + z = 2\sqrt{3} + 1$ then prove that :

$$\frac{1}{x} + \frac{1}{y^2} + \frac{1}{z^3} \geq \frac{4\sqrt{3} + 9}{9}$$

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By AM − GM inequality, we have :

$$\frac{1}{x} + \frac{1}{x} + \frac{1}{x} + x^3 \geq 4 \;\Rightarrow\; \frac{1}{x} \geq \frac{4}{3} - \frac{x^3}{3} \quad (1)$$

$$\frac{1}{y^2} + \frac{y^2}{3} \geq \frac{2}{\sqrt{3}} \;\Rightarrow\; \frac{1}{y^2} \geq \frac{2\sqrt{3}}{3} - \frac{y^2}{3} \quad (2)$$

$$\frac{1}{z^3} + \frac{z}{9} + \frac{z}{9} + \frac{z}{9} \geq \frac{4}{\sqrt[4]{9^3}} \;\Rightarrow\; \frac{1}{z^3} \geq \frac{4\sqrt{3}}{9} - \frac{z}{3} \quad (3)$$

Adding (1), (2) and (3), we get

$$\frac{1}{x} + \frac{1}{y^2} + \frac{1}{z^3} \geq \frac{10\sqrt{3} + 12}{9} - \frac{1}{3}\left(x^3 + y^2 + z\right) = \frac{10\sqrt{3} + 12}{9} - \frac{2\sqrt{3} + 1}{3} = \frac{4\sqrt{3} + 9}{9}.$$

Equality holds iff $x = 1, y = \sqrt[4]{3}, z = \sqrt{3}$.