ROMANIAN MATHEMATICAL MAGAZINE

If $0 < x < y < 4$, *then prove that* :

$$\ln\frac{x(4-y)}{y(4-x)} < x - y$$

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$$\ln\frac{x(4-y)}{y(4-x)} < x - y \Leftrightarrow \ln x + \ln(4-y) - \ln y - \ln(4-x) < x - y$$

$$\Leftrightarrow \ln y - y - \ln(4-y) > \ln x - x - \ln(4-x) \Leftrightarrow f(y) - f(x) > 0$$

$$\big(f(t) = \ln t - t - \ln(4-t)\ \forall\, t \in (0,4)\big) \overset{\text{via MVT}}{\Leftrightarrow} (y-x).f'(c) > 0$$

$$(0 < x < c < y < 4) \Leftrightarrow (y-x)\left(\frac{1}{c} + \frac{1}{4-c} - 1\right) > 0 \to \text{true} \because (y-x) > 0$$

$$\textit{and}\ \frac{1}{c} + \frac{1}{4-c} \overset{\text{Bergstrom}}{\geq} \frac{4}{c+4-c} = 1 \Rightarrow \frac{1}{c} + \frac{1}{4-c} - 1 > 0$$

$$\therefore \ln\frac{x(4-y)}{y(4-x)} < x - y \text{ for } 0 < x < y < 4\ (QED)$$