ROMANIAN MATHEMATICAL MAGAZINE

If $0 < x \le y \le z \le 4$ and $xyz = 1$, then prove that:

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z}}\le\sqrt{5}$$

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$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z}}=$$

$$=\frac{1}{2\sqrt{1+x^2}}+\frac{1}{2\sqrt{1+x^2}}+\frac{1}{2\sqrt{1+y^2}}+\frac{1}{2\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z}}$$

$$\overset{CBS}{\le}\sqrt{5}.\sqrt{\frac{1}{4(1+x^2)}+\frac{1}{4(1+x^2)}+\frac{1}{4(1+y^2)}+\frac{1}{4(1+y^2)}+\frac{1}{1+z}}\overset{?}{\le}\sqrt{5}$$

$$\Leftrightarrow\frac{(1+z)(1+y^2)+(1+z)(1+x^2)+2(1+x^2)(1+y^2)}{2(1+x^2)(1+y^2)(1+z)}\overset{?}{\le}1\Leftrightarrow$$

$$2x^2y^2z+z(x^2+y^2)-x^2-y^2-2\overset{?}{\ge}0\overset{z=\frac{1}{xy}}{\Leftrightarrow}2xy+\frac{x^2+y^2}{xy}-(x^2+y^2)-2\overset{?}{\ge}0$$

$$\Leftrightarrow 2x^2y^2+x^2+y^2-xy(x^2+y^2)-2xy\overset{?}{\ge}0$$

$$\Leftrightarrow(x^2+y^2-2xy)-xy(x^2+y^2-2xy)\overset{?}{\ge}0$$

$$\Leftrightarrow(x-y)^2(1-xy)\overset{?}{\ge}0\overset{xy=\frac{1}{z}}{\Leftrightarrow}(x-y)^2\left(1-\frac{1}{z}\right)\overset{?}{\ge}0\Leftrightarrow(z-1)(x-y)^2\overset{?}{\ge}0$$

$$\rightarrow \text{true} \because z\ge x,y\Rightarrow z^2\ge xy\Rightarrow z^3\ge xyz=1\Rightarrow z-1\ge 0$$

$$\therefore\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z}}\le\sqrt{5}\text{ for } 0<x\le y\le z\le 4\wedge xyz=1,$$

with equality iff $x=y$ and $\frac{1}{2\sqrt{1+x^2}}=\frac{1}{\sqrt{1+z}}=\frac{1}{\sqrt{1+\frac{1}{x^2}}}\left(\because z=\frac{1}{xy}=\frac{1}{x^2}\right)$

$$\Rightarrow x=y=\frac{1}{2}\Rightarrow z=\frac{1}{\left(\frac{1}{2}\right)^2}=4\therefore "=" \text{ iff }\left(x=y=\frac{1}{2},z=4\right)\text{ (QED)}$$