ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x^2 + y^2 + z^2 = 1$, then prove that :

$$\frac{1}{\sqrt{x^2+xy}} + \frac{1}{\sqrt{y^2+xy}} + \frac{2\sqrt{3}}{1+z} \geq \frac{8\sqrt{3}}{3}$$

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$$\frac{1}{\sqrt{x^2+xy}} + \frac{1}{\sqrt{y^2+xy}} \overset{\text{Radon}}{\geq} \frac{2^{\frac{3}{2}}}{\sqrt{x^2+xy+y^2+xy}} = \frac{\sqrt{8}}{x+y}$$

$$\overset{\text{Radon}}{\geq} \frac{\sqrt{8}}{\sqrt{2(x^2+y^2)}} \overset{x^2+y^2+z^2=1}{=} \frac{2}{\sqrt{1-z^2}}$$

$$\Rightarrow \frac{1}{\sqrt{x^2+xy}} + \frac{1}{\sqrt{y^2+xy}} + \frac{2\sqrt{3}}{1+z} \overset{(*)}{\geq} \frac{2}{\sqrt{1-z^2}} + \frac{2\sqrt{3}}{1+z}$$

$$\text{Let } f(z) = \frac{1}{\sqrt{1-z^2}} + \frac{\sqrt{3}}{1+z} \ \forall z \in (0,1) \text{ and then :}$$

$$f'(z) = \frac{z}{(1-z^2)^{\frac{3}{2}}} - \frac{\sqrt{3}}{(z+1)^2} \to (1)$$

$$\text{Now, when : } \frac{z}{(1-z^2)^{\frac{3}{2}}} - \frac{\sqrt{3}}{(z+1)^2} = 0, \text{ then : } \frac{z^2}{(1-z)^3(1+z)^3} = \frac{3}{(z+1)^4}$$

$$\Rightarrow 4z^3 - 8z^2 + 9z - 3 = 0 \Rightarrow (2z-1)(2z^2-3z+3) = 0 \text{ and}$$

$$\because \text{discriminant of } (2z^2-3z+3) = 9 - 24 < 0 \therefore 2z^2 - 3z + 3 > 0 \text{ and so,}$$

$$\frac{z}{(1-z^2)^{\frac{3}{2}}} - \frac{\sqrt{3}}{(z+1)^2} = 0 \Rightarrow z = \frac{1}{2} \therefore f'\left(\frac{1}{2}\right) = 0 \text{ and}$$

$$f''\left(\frac{1}{2}\right) = \left.\frac{1+3z^2}{(1-z^2)^{\frac{3}{2}}} + \frac{2\sqrt{3}}{(z+1)^3}\right|_{z=\frac{1}{2}} > 0 \therefore f(z) \text{ attains a minima at } z = \frac{1}{2}$$

$$\text{and} \because f''(z) = \frac{1+3z^2}{(1-z^2)^{\frac{3}{2}}} + \frac{2\sqrt{3}}{(z+1)^3} > 0 \ \forall z \in (0,1) \therefore f'(z) \text{ is } \uparrow \text{ on } (0,1)$$

$$\therefore f'(z) = 0 \text{ has a unique root, it being } z = \frac{1}{2} \text{ and so, } f(z) \text{ never attains a maxima}$$

$$\Rightarrow f(z) \geq f\left(\frac{1}{2}\right) = \left.\frac{1}{\sqrt{1-z^2}} + \frac{\sqrt{3}}{1+z}\right|_{z=\frac{1}{2}} = \frac{4\sqrt{3}}{3} \Rightarrow \frac{2}{\sqrt{1-z^2}} + \frac{2\sqrt{3}}{1+z} \overset{(**)}{\geq} \frac{8\sqrt{3}}{3}$$

$$\therefore (*), (**) \Rightarrow \frac{1}{\sqrt{x^2+xy}} + \frac{1}{\sqrt{y^2+xy}} + \frac{2\sqrt{3}}{1+z} \geq \frac{8\sqrt{3}}{3}$$

$$\forall x, y, z > 0 \,\middle|\, x^2 + y^2 + z^2 = 1, '' = '' \text{ iff } \left(x = y = \frac{\sqrt{6}}{4}, z = \frac{1}{2}\right) \text{ (QED)}$$