ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ then prove that :

$$\frac{(a-\mathrm{b})^2}{4(2a+\mathrm{b})^2}+\frac{(2a+\mathrm{b})(7a+5\mathrm{b})}{(a+\mathrm{b})(3a+\mathrm{b})}\le\frac{3(2a+\mathrm{b})^2}{2a(a+2\mathrm{b})}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{3(2a+\mathrm{b})^2}{2a(a+2\mathrm{b})}-\frac{(2a+\mathrm{b})(7a+5\mathrm{b})}{(a+\mathrm{b})(3a+\mathrm{b})}=$$

$$=(2a+\mathrm{b})\left(\frac{3(2a+\mathrm{b})(a+\mathrm{b})(3a+\mathrm{b})-2a(a+2\mathrm{b})(7a+5\mathrm{b})}{2a(a+2\mathrm{b})(a+\mathrm{b})(3a+\mathrm{b})}\right)=$$

$$=(2a+\mathrm{b})\left(\frac{4a^3-5a^2-2a\mathrm{b}^2+3\mathrm{b}^3}{2a(a+2\mathrm{b})(a+\mathrm{b})(3a+\mathrm{b})}\right)=\frac{(2a+\mathrm{b})(4a+3\mathrm{b})(a-\mathrm{b})^2}{2a(a+2\mathrm{b})(a+\mathrm{b})(3a+\mathrm{b})}=$$

$$=\frac{(a+a+\mathrm{b})^2(2a+\mathrm{b})(4a+3\mathrm{b})(a-\mathrm{b})^2}{2a(a+2\mathrm{b})(a+\mathrm{b})(3a+\mathrm{b})(2a+\mathrm{b})^2}\overset{\text{AM-GM}}{\ge}$$

$$\ge\frac{4a(a+\mathrm{b})(2a+\mathrm{b})(4a+3\mathrm{b})(a-\mathrm{b})^2}{2a(a+2\mathrm{b})(a+\mathrm{b})(3a+\mathrm{b})(2a+\mathrm{b})^2}=$$

$$=\left(1+\frac{8(2a+\mathrm{b})(4a+3\mathrm{b})}{(a+2\mathrm{b})(3a+\mathrm{b})}-1\right)\cdot\frac{(a-\mathrm{b})^2}{4(2a+\mathrm{b})^2}=$$

$$=\left(1+\frac{61a^2+73a\mathrm{b}+22\mathrm{b}^2}{(a+2\mathrm{b})(3a+\mathrm{b})}\right)\cdot\frac{(a-\mathrm{b})^2}{4(2a+\mathrm{b})^2}\ge\frac{(a-\mathrm{b})^2}{4(2a+\mathrm{b})^2}$$

$$\therefore\frac{(a-\mathrm{b})^2}{4(2a+\mathrm{b})^2}+\frac{(2a+\mathrm{b})(7a+5\mathrm{b})}{(a+\mathrm{b})(3a+\mathrm{b})}\le\frac{3(2a+\mathrm{b})^2}{2a(a+2\mathrm{b})}\ \forall\, a,\mathrm{b}>0,''='' \text{ iff } a=\mathrm{b}\ (\text{QED})$$