ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b} > 0$ *and* $\mathrm{n} \in \mathbb{N}^*$ then :

$$e^{\frac{1}{\sqrt[n]{a}}} + b\ln b \ge b + \frac{b}{\sqrt[n]{a}}$$

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$$\frac{1}{\sqrt[n]{a}} = \ln e^{\frac{1}{\sqrt[n]{a}}} = \ln u \left(say;\ u = e^{\frac{1}{\sqrt[n]{a}}}\right) \text{ and then : } e^{\frac{1}{\sqrt[n]{a}}} + b\ln b \ge b + \frac{b}{\sqrt[n]{a}}$$

$$\Leftrightarrow u + b\ln b \ge b + b.\ln u \Leftrightarrow u - b \boxed{\overset{(*)}{\ge}} b(\ln u - \ln b) \text{ and if } e^{\frac{1}{\sqrt[n]{a}}} = u = b, \text{then,}$$

$$\text{LHS of } (*) = \text{RHS of } (*) = 0 \text{ and now we focus on } u \ne b$$

$$\boxed{Case\ 1}\ u > b \text{ and since the derivative of } \ln x \text{ w.r.t. } x \text{ is } \frac{1}{x},$$

$$\therefore \ln u - \ln b \overset{\text{MVT}}{=} (u-b).\frac{1}{\xi}\ (\xi \,|\, b < \xi < u) \therefore (*) \Leftrightarrow u - b \overset{?}{\ge} (u-b).\frac{b}{\xi}$$

$$\Leftrightarrow (u-b)\left(\frac{\xi - b}{\xi}\right) \overset{?}{\ge} 0 \to \text{true} \because 0 < b < \xi < u \Rightarrow \xi > 0 \text{ and } (u-b), (\xi - b) > 0$$

$$\Rightarrow (*) \text{ is true}$$

$$\boxed{Case\ 2}\ b > u \text{ and } (*) \Leftrightarrow b(\ln b - \ln u) \boxed{\overset{(**)}{\ge}} b - u \text{ and again, } b(\ln b - \ln u)$$

$$\overset{\text{MVT}}{=} \frac{b(b-u)}{\eta}\ (\eta \,|\, u < \eta < b) \overset{?}{\ge} b - u \Leftrightarrow (b-u)\left(\frac{b-\eta}{\eta}\right) \overset{?}{\ge} 0 \to \text{true}$$

$$\because 0 < u < \eta < b \Rightarrow \eta > 0 \text{ and } (b-u), (b-\eta) > 0 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \text{combining both cases, } e^{\frac{1}{\sqrt[n]{a}}} + b\ln b \ge b + \frac{b}{\sqrt[n]{a}}\ \forall\, a, b > 0 \text{ and } n \in \mathbb{N}^*,$$

$$'' ='' \text{ iff } e^{\frac{1}{\sqrt[n]{a}}} = b\ (\text{QED})$$