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If $n \geq 2$ then prove

$$\sqrt{n^2-1}+\sqrt{n^2-4}+\sqrt{n^2-9}+\dots+\sqrt{2n-1} > 0{,}785n^2-n$$

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$$LHS=\sqrt{n^2-1}+\sqrt{n^2-4}+\sqrt{n^2-9}+\dots+\sqrt{2n-1}=$$

$$=\sqrt{n^2-1^2}+\sqrt{n^2-2^2}+\sqrt{n^2-3^2}+\dots+\sqrt{n^2-(n-1)^2}$$

$$LHS=\sum_{k=1}^{n-1}\sqrt{n^2-k^2} > 0{,}785n^2-n$$

Asymptotic Evaluation of the Summation using the Integral Test:

$$LHS=\sum_{k=1}^{n-1}\sqrt{n^2-k^2} > \int_1^n \sqrt{n^2-x^2}dx \overset{x\to n\sin(x)}{\triangleq} \left[\frac{x\sqrt{n^2-x^2}}{2}+\frac{n^2}{2}\sin^{-1}\left(\frac{x}{n}\right)\right]_1^n=$$

$$=\frac{\pi}{4}n^2-\frac{\sqrt{n^2-1}}{2}-\frac{n^2}{2}\sin^{-1}\left(\frac{1}{n}\right)=\frac{\pi n^2}{4}-\left(\frac{\sqrt{n^2-1}}{2}+\frac{n^2}{2}\sin^{-1}\left(\frac{1}{n}\right)\right)=\frac{\pi n^2}{4}-I(n)$$

$$I(n)=\frac{\sqrt{n^2-1}}{2}+\frac{n^2}{2}\sin^{-1}\left(\frac{1}{n}\right) \to \frac{\sqrt{n^2-1}}{2}=\frac{n}{2}\left(1-\frac{1}{n^2}\right)^{\frac{1}{2}}$$

$$Binmial\ series: (1+y)^z \approx 1+yz+\frac{z(z-1)}{2!}y^2+\cdots$$

$$\frac{\sqrt{n^2-1}}{2}=\frac{n}{2}\left(1-\frac{1}{n^2}\right)^{\frac{1}{2}} \approx \frac{n}{2}\left(1-\frac{1}{2n^2}-\frac{1}{8n^4}+O\left(\frac{1}{n^6}\right)\right) \sim \frac{n}{2}-\frac{1}{4n}+O\left(\frac{1}{n^3}\right)$$

$$Taylor\ series\ of\ \arcsin(x)=\sum_{k=0}^{\infty}\frac{(2k)!}{4^k(k!)^2(2k+1)}x^{2k+1}\ \ as\ |x|\leq 1$$

$$\left|\frac{1}{n}\right| < 1\ as\ n\geq 2\ \ \arcsin\left(\frac{1}{n}\right)=\frac{1}{n}+\frac{1}{6n^3}+\sum_{k=2}^{\infty}\frac{(2k)!\left(\frac{1}{n}\right)^{2k+1}}{4^k(k!)^2(2k+1)} \sim \frac{1}{n}+\frac{1}{6n^3}+O\left(\frac{1}{n^5}\right)$$

$$\to \frac{n^2}{2}\left(\frac{1}{n}+\frac{1}{6n^3}+O\left(\frac{1}{n^5}\right)\right) \sim \frac{n}{2}+\frac{1}{12n}+O\left(\frac{1}{n^3}\right)$$

$$I(n) \sim n + \frac{1}{12n} - \frac{1}{4n} + O\left(\frac{1}{n^3}\right) \rightarrow\ I(n) \sim n - \frac{1}{6n} + O\left(\frac{1}{n^3}\right) \rightarrow\ I(n) \leq n$$

$$we\ know\ \frac{\pi}{4} \approx 0,78539 \ldots > 0,785 \quad LHS = \frac{\pi n^2}{4} - I(n) > \frac{\pi n^2}{4} - n > 0,785n^2 - n$$

$$Therefore\ \sqrt{n^2-1} + \sqrt{n^2-4} + \sqrt{n^2-9} + \cdots + \sqrt{2n-1} > 0,785n^2 - n\ as\ n \geq 2$$