ROMANIAN MATHEMATICAL MAGAZINE

If $0 < y < x, n \in \mathbb{N}, n \geq 2$ then:

$$\frac{x-y}{n\sqrt[n]{x^{n-1}}} < \sqrt[n]{x} - \sqrt[n]{y} < \frac{x-y}{n\sqrt[n]{y^{n-1}}}$$

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Solution by Daniel Sitaru-Romania

Let be $f: [y, x] \to \mathbb{R}, f(t) = \sqrt[n]{t},\ f'(t) = \frac{1}{n\sqrt[n]{t^{n-1}}}$

By Lagrange's mean value theorem:

$$\exists c \in (y, x), f(x) - f(y) = \frac{1}{n\sqrt[n]{c^{n-1}}}(x-y)$$

$$\sqrt[n]{x} - \sqrt[n]{y} = \frac{x-y}{n\sqrt[n]{c^{n-1}}} \qquad (1)$$

$$c \in (y, x) \Rightarrow y < c < x \Rightarrow y^{n-1} < c^{n-1} < x^{n-1} \Rightarrow$$

$$\sqrt[n]{y^{n-1}} < \sqrt[n]{c^{n-1}} < \sqrt[n]{x^{n-1}} \Rightarrow n\sqrt[n]{y^{n-1}} < n\sqrt[n]{c^{n-1}} < n\sqrt[n]{x^{n-1}} \Rightarrow$$

$$\frac{1}{n\sqrt[n]{x^{n-1}}} < \frac{1}{n\sqrt[n]{c^{n-1}}} < \frac{1}{n\sqrt[n]{y^{n-1}}} \Rightarrow \frac{x-y}{n\sqrt[n]{x^{n-1}}} < \frac{x-y}{n\sqrt[n]{c^{n-1}}} < \frac{x-y}{n\sqrt[n]{y^{n-1}}} \overset{(1)}{\Rightarrow}$$

$$\frac{x-y}{n\sqrt[n]{x^{n-1}}} < \sqrt[n]{x} - \sqrt[n]{y} < \frac{x-y}{n\sqrt[n]{y^{n-1}}}$$