ROMANIAN MATHEMATICAL MAGAZINE

$$If\ a,b,c,d>0\ \&\frac{a-d}{a+d}+\frac{b-d}{b+d}+\frac{c-d}{c+d}=0$$

$$then\ prove\ that\ \left(d-\sqrt[3]{abc}\right)\left(d-\frac{ab+bc+ca}{a+b+c}\right)\le 0$$

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Solution by Tapas Das-India

$$abc-d^3=\left(\sqrt[3]{abc}\right)^3-d^3=\left(\sqrt[3]{abc}-d\right)\left(\left(\sqrt[3]{abc}\right)^2+d\sqrt[3]{abc}+d^2\right)\ (1)$$

$$\frac{a-d}{a+d}+\frac{b-d}{b+d}+\frac{c-d}{c+d}=0$$

$$or,(a-d)(b+d)(c+d)+(b-d)(c+d)(a+d)+(c-d)(a+d)(b+d)=0$$

$$3abc+d(ab+bc+ca)-d^2(a+b+c)-3d^3=0$$

$$or,(ab+bc+ca)-d(a+b+c)=\frac{3(d^3-abc)}{d}\ (2)$$

$$now\ \left(d-\sqrt[3]{abc}\right)\left(d-\frac{ab+bc+ca}{a+b+c}\right)=\frac{\left(d-\sqrt[3]{abc}\right)}{a+b+c}\left(d(a+b+c)-(ab+bc+ca)\right)$$

$$\overset{(2)}{=}\frac{\left(d-\sqrt[3]{abc}\right)}{a+b+c}\cdot\frac{3(abc-d^3)}{d}\overset{(1)}{=}\frac{\left(d-\sqrt[3]{abc}\right)3}{a+b+c}\frac{\left(\sqrt[3]{abc}-d\right)\left(\left(\sqrt[3]{abc}\right)^2+d\sqrt[3]{abc}+d^2\right)}{d}$$

$$=-\frac{3\left(\sqrt[3]{abc}-d\right)^2\left(\left(\sqrt[3]{abc}\right)^2+d\sqrt[3]{abc}+d^2\right)}{d}\le 0\ as\ d>0$$

$$Equaliy\ holds\ for\ (a,b,c)=\left(d,dt,\frac{d}{t}\right)\ for\ t>0\ and\ in\ any\ permutation.$$