ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c,d>0$ then:

$$3(a^2+b^2+c^2+d^2)+4\sqrt{abcd}\geq(a+b+c+d)^2+2\left(\sqrt{a}-\sqrt{b}\right)^2\left(\sqrt{c}-\sqrt{d}\right)^2$$

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$\boldsymbol{Identity}$:

$$(p^4-p^2q^2+q^4)=(p^2-pq+q^2)^2+2pq(p-q)^2$$

$\boldsymbol{Proof}$:

$$(p^2-pq+q^2)^2+2pq(p-q)^2=$$
$$=(p^4+p^2q^2+q^4+2p^2q^2-2p^3q-2q^3p)+2pq(p^2-2pq+q^2)=p^4-p^2q^2+q^4$$

$$\boldsymbol{Let\ a=x^2,b=y^2,c=z^2,d=w^2\ then\ given\ ineqaliy\ can\ be\ written\ as}$$
$$3(x^4+y^4+z^4+w^4)+4xyzw\geq(x^2+y^2+z^2+w^2)^2+2(x-y)^2(y-z)^2$$
$$3\left(x^4+y^4+z^4+w^4\right)+4xyzw\geq$$
$$\geq\left(x^4+y^4+z^4+w^4\right)+2\left(x^2+y^2\right)\left(w^2+z^2\right)+2\left(x^2y^2+w^2z^2\right)+2(x-y)^2(y-z)^2$$

$$2\sum x^4+4xyzw\geq$$
$$\geq 2(x^2y^2+w^2z^2)+2(x^2+y^2)(w^2+z^2)+2(x^2-2xy+y^2)(w^2-2wz+z^2)$$

$$\sum x^4+2xyzw\geq$$
$$\geq(x^2y^2+w^2z^2)+(x^2+y^2)(w^2+z^2)+(x^2-2xy+y^2)(w^2-2wz+z^2)$$
$$=(x^2y^2+w^2z^2)+(x^2+y^2)(w^2+z^2)+(x^2+y^2)(w^2+z^2)-2wz(x^2+y^2)$$
$$-2xy(w^2+z^2)+4xyzw$$

$$(x^4-x^2y^2+y^4)+(z^4-z^2w^2+w^4)\geq$$
$$\geq 2\left((x^2+y^2)(w^2+z^2)-wz(x^2+y^2)-xy(w^2+z^2)+xyzw\right)$$

$$(x^4-x^2y^2+y^4)+(z^4-z^2w^2+w^4)\geq 2(x^2-xy+y^2)(w^2-wz+z^2)$$
$$(x^2-xy+y^2)^2+2xy(x-y)^2+(z^2-wz+w^2)^2+2wz(w-z)^2 \overset{Identity}{\geq}$$
$$\geq 2(x^2-xy+y^2)(w^2-wz+z^2)$$
$$or,A^2+2xy(x-y)^2+B^2+2wz(w-z)^2\geq 2AB\ or,$$
$$(A-B)^2+2xy(x-y)^2+2wz(w-z)^2\geq 0\ True$$
$$(A=x^2-xy+y^2,B=w^2-wz+z^2)$$

$$\boldsymbol{Equality\ holds\ for\ a=b=c=d.}$$